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# Finding whether a point is inside a polygon

## And Its Application To Forbidden Regions*

**Zvi Drezner**
*California State University-Fullerton*

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**Abstract**

In many location problems, the location of the facility is restricted to a certain region. One would not locate a hospital in the middle of the ocean, or a shipyard not on the beach. Usually, such constraints can be expressed as a requirement that the location of the facility must be inside a polygon (which might be non-convex). When a polygon is convex, a point is inside the polygon if and only if all the constraints that define the sides of the polygon are fulfilled. If the polygon is not convex, this is no longer correct. In this note we devise one constraint that will determine whether a given point is inside a polygon or not. This constraint can be used for both convex or non-convex polygons. The method is demonstrated for solving location problems with forbidden regions.

## 1. Introduction

The issue of finding whether a point is inside a polygon has been treated by geologists [1,5,9,12,16], physicists [15,19], and computer scientists [4,11,17,18]. Two main methods are proposed for this problem. One way is to count how many times a ray which originates at the point intersects with the sides of the polygon. An even number indicates that the point is outside the polygon while an odd number indicates that the point is inside. A second way is to add up the angles subtended from the point for all the sides of the polygon. If the sum is zero, then the point is outside the polygon, and if the sum is $\pm 2\pi$ then the point is inside. To the best of my knowledge, there is no paper in the operations research literature that deals with this issue. The purpose of this paper is to turn the attention of operations researchers to this problem and its applications in spatial analysis.

The issue is an important to many fields of operations research. Assume that one needs to define a constraint to find out whether a certain site is in the U.S. or not. The boundary of the U.S. can be approximated by a polygon, but the polygon is not convex and therefore one cannot incorporate such a definition in a nonlinear formulation. The same question can be asked about any desired area such as states, metropolitan areas, etc. In a production setting in a plant, a facility may need to be located. The location must be restricted to the aisles or off the aisles. The feasible area may have an irregular pattern. One constraint suffices to figure out whether a point is in an aisle or off the aisles. This is an important issue in many location, spatial, and production models [7,8,10,13,14].

* Part of this research was done while the author was on sabbatical leave at the Department of Management, The Hong Kong University of Science and Technology, Kowloon, Hong Kong.

In this paper we show that one constraint that determines whether a point is inside a polygon or not can be constructed, based on the sum of angles approach. Even though the first method (counting the number of times that a ray intersects the polygon) seems to perform better in experiments [5,15,16], it cannot be formulated as a constraint. This constraint can be incorporated into a nonlinear programming formulation. It may also simplify formulations for convex polygons with many sides. Instead of having many constraints (as many as the number of sides of the polygon), one can use the one constraint suggested here. It should be noted, however, that this constraint is not linear.

One constraint can be constructed for any polygonal area, whether it has only one connected boundary, or there are "holes" in the area, or the area is a disjoint union of areas. The method is demonstrated by solving a location problem in the presence of forbidden regions. For a literature review of this problem see [2]. This method is especially fitted for a solution by Excel.

## 2. Constructing the constraint

We first consider a polygon that has one connected boundary. Such a polygon is defined by a sorted list of vertices $(x_i, y_i)$ for $i = 1,..,n.$ The boundary of the polygon is defined by the segments connecting successive vertices. We also assume that $(x_1, y_1) = (x_n, y_n)$, which means that the boundary is connected, and that the boundary segments do not cross each other. The order of the vertices in the list can be reversed and we get the same polygon. There are two possible sequences of vertices. We intuitively require that the vertices will be sorted "counterclockwise." Another way to determine the correct order is to perform the following test. A starting vertex is selected. Two vertices are its immediate neighbors. The correct direction is to the neighbor for which the feasible region is "on your right" when moving along the side connecting the two vertices. If the feasible region is on your "left," the other vertex should be the second in the list. Once the direction is determined, all other vertices are automatically determined until the polygon is closed by returning to the first vertex. This rule is also useful for determining the direction of vertices defining a "hole" in the feasible region.

Let $(x, y)$ be a point in the plane; we need to decide whether it is inside the polygon or not. For the following analysis we assume that point $(x, y)$ is not on the border of the polygon (including the vertices). The idea for the constraint is as follows. Consider two rays originating at $(x, y)$. One ray is passing through vertex $i$ and the other passing through vertex $i+1$. Calculate the directed angle between these two rays. This angle is a directional angle (from vertex $i$ to vertex $i+1$) and is always between -π and π. Let this directional angle between vertex $i$ and vertex $i+1$ be $\theta_i$ for $i = 1,..,n-1$. When all these angles are added up, the sum is $2\pi$ (or -2π, if the vertices are ordered in the "wrong" order) if the point is inside the polygon. The sum is zero if the point is outside the polygon. This property can be rigorously proved as follows (it is believed that this is a new proof):

**Theorem 1**: The sum of angles from a point inside the polygon is $2\pi$, and 0 from a point outside a polygon.

**Proof**: We prove the theorem based on a fundamental property of functions in a complex variable. Analytical complex functions may have poles, and each pole defines a residue [3]. A pole is a point where the function has an infinite value, and the residue is the coefficient of $\frac{1}{z}$ in the expansion of the function to power series. The following theorem is the basis of complex function analysis [3]: $\oint f(z)dz = 2\pi i \sum residues$, where the residues are added up for all the poles inside the contour of integration. For convenience, assume that the point checked for being inside the polygon is located at the origin. Consider the function $f(z)=\frac{1}{iz}$. This function has one pole at the origin with a residue of $\frac{1}{i}$. Therefore, the contour integral along the polygon is equal to 2π if the origin is inside the polygon, and to zero if the origin is outside the polygon. To complete the proof, we need to show that the contour integral is equal to the sum of the angles formed by each side of the polygon. Indeed, consider the integral between two consecutive vertices $z_1$ and $z_2$ defining a segment that does not pass through the origin. By polar coordinates $z=re^{i\theta}$, and therefore $dz=ire^{i\theta}d\theta+e^{i\theta}dr=izd\theta+\frac{dr}{r}$. Thus,

$$\int_{z_1}^{z_2} f(z)dz=\int_{z_1}^{z_2}\frac{dz}{iz}=\int_{\theta_1}^{\theta_2} d\theta-i\int_{r_1}^{r_2}\frac{dr}{r}=\theta_2-\theta_1-i[\ln r_2-\ln r_1].$$

Since the contour integral starts and ends at the same radius, the imaginary terms add up to zero, while the real terms add up to the sum of the angles. The theorem follows.
The constraint is defined by "the sum of all the angles $\theta_i$, for $i=1,..,n-1$ is greater than or equal to π." Actually, any number between 0 and 2π can serve as the right-hand side.

## 3. Calculating an angle

There are many possible ways to calculate the directed angle between two points as seen from a third one. We found the following approach useful. Directional angles are invariant to translation and rotation. Change the system of coordinates such that the point $(x,y)$ is at (0, 0). We calculate the angle from a vertex $(a,b)$ to a vertex $(c,d)$. It is assumed that none of these points is located at the origin. The system of coordinates is rotated such that point $(a,b)$ is located on the positive x-axis. The point $(c,d)$ is transformed to: $\left(\frac{ac+bd}{\sqrt{a^2+b^2}},\frac{ad-bc}{\sqrt{a^2+b^2}}\right)$. Let $\theta$ be the directed angle from $(a,b)$ to $(c,d)$. We have:

$$\cos\theta=\frac{ac+bd}{\sqrt{(ac+bd)^2+(ad-bc)^2}};\quad \sin\theta=\frac{ad-bc}{\sqrt{(ac+bd)^2+(ad-bc)^2}}.$$

Since the function $ar\cos(x)$ gives a result in [0,π] it seems easier to use. So we use

$$\theta=\operatorname{sgn}(ad-bc)\cdot\arccos\left(\frac{ac+bd}{\sqrt{(ac+bd)^2+(ad-bc)^2}}\right)$$

where $\operatorname{sgn}(x)$ is the sign function ($\operatorname{sgn}(x)=1$ for $x\geq 0$ and -1 for $x<0$).

In order to have $\theta$ defined for every point in the plane, we add a small $\varepsilon>0$ to denominators that may vanish. This will cause the boundary of the polygon to be outside the polygon. For a small enough $\varepsilon>0$, the error will not exceed π, which is the maximum allowed error when π is used as the right-hand side.

Define:

$$\begin{aligned} a_i &= (x_i - x)(x_{i+1} - x) + (y_i - y)(y_{i+1} - y) \\ b_i &= (x_i - x)(y_{i+1} - y) - (y_i - y)(x_{i+1} - x) \end{aligned} \quad \text{for } i = 1,\dots,n-1.$$

The constraint (for a point $(x, y)$ to be inside a polygon), for a small $\varepsilon > 0$, is:

$$\sum_{i=1}^{n-1} \frac{b_i}{|b_i| + \varepsilon} \arccos\left( \frac{a_i}{\sqrt{a_i^2 + b_i^2 + \varepsilon^2}} \right) \geq \pi \tag{1}$$

## 4. Notes

1. If the point is inside the polygon, the sum in (1) is $2\pi$ if the order of vertices is counterclockwise, or $-2\pi$ otherwise. However, it is the same value for all interior points. If one is unsure whether the vertices are ordered correctly, one interior point can be checked by (1). If the sum is $-2\pi$, the order of the vertices should be reversed. As explained above, if the order of vertices is correct, the feasible region should be "on your right" when moving from one vertex to the next one. An alternative way to determine the right order for the vertices is using the calculation of the area of a polygon. In [6], the area $S$ of a general polygon (not necessarily convex) is calculated as: $S = \frac{1}{2}\sum_{i=1}^{n-1} x_i(y_{i+1} - y_{i-1})$, where $(x_0, y_0) = (x_n, y_n)$. If the vertices are ordered in a counterclockwise order the calculated area $S$ is positive. If $S$ is negative, the order of the vertices should be reversed.
2. If the area is a union of disjoint polygons, consider the accumulated sums by equation (1) for all polygons separately. The sum for each individual polygon is either 0 or $2\pi$. Therefore, the total of all sums is zero if the point is outside all the polygons and is $2\pi$ if the point is inside any of the polygons.
3. If there are "holes" in the area, a point inside the area must be outside the union of the holes. Let $SS$ be the sum of (1) for the outside boundary minus all the individual sums for the holes. For an interior point, $SS = 2\pi$, and for exterior points, $SS \leq 0$. Therefore, $SS \geq \pi$ is the desired constraint.

## 5. Application to forbidden regions

Consider any location problem (say, the Weber problem) with forbidden regions which can be represented by polygons (each forbidden polygon may be either convex or not). Forbidding a facility to be located inside any of these regions can be formulated as one constraint. A local minimum for such a problem can be found, for example, by Excel as a non-linear programming problem with an objective function and one constraint. Note that only one constraint is needed regardless of the number of forbidden regions. By using other methods, representing such constraints in a standard non-linear programming fashion is not possible even if all forbidden regions are convex, because the collection of constraints for each forbidden polygon needs to be compiled with Boolean "or" constraints.

## 6. Numerical examples

### 6.1 A contrived polygon example

A contrived polygon shown in Figure 1 in the square $0 \le x, y \le 1$ was selected. The area of the polygon is 0.58. One thousand points were randomly generated in the square and 596 of them were found inside the polygon by the evaluation of (1). This number is about one standard deviation over the expected number of 580. These 596 points are depicted in Figure 2. Clearly, all of them are inside the polygon.

### 6.2 A forbidden regions example

Consider a location problem in the Middle East. Demand is generated in ten cities shown in Figure 3. We need to find a location for a central emergency facility such as an organ bank. The objective is therefore the minimax objective, which is to minimize the maximum travel time to all cities. We assume transportation by light planes, which entails Euclidean distances. The region contains a few bodies of water which are not suitable for the location of the planned facility and one island (Cyprus). The bodies of water are forbidden regions. The polygon is generated as follows (see Figure 3): The vertices are selected clockwise on the southern shore of Turkey surrounding the Mediterranean Sea, which is actually counterclockwise around the feasible region. The polygon continues to exclude the Red Sea and returns to the first vertex. The last vertex, vertex 43, is the same as vertex 1. A different polygon is designed for Cyprus and is ordered in a counterclockwise direction because it is surrounding a feasible area. This polygon has 12 vertices with the last one being identical to the first. There is only one constraint combining the sum of angles for both polygons. Additional forbidden regions could be added (such as the Dead Sea, which we did not add because it would have cluttered the map), and they should be ordered clockwise. We covered an area from $31^\circ$ to $38^\circ$ East and $26^\circ$ to $38^\circ$ North. Each degree of longitude was calculates as 95 km and each degree of latitude as 111 km. The point (31E,26N) was used as the origin.

The minimax one-center problem was coded in Excel using the solver option. The unconstrained version of the problem was found in the Mediterranean Sea as depicted in Figure 3. Its coordinates are (261,814) which is equidistant from Cairo, Eilat, and Adana with a maximum distance of 435 km. Halab, whose distance is 428 km, is close to the maximum distance. This solution is not feasible because it is in the water. We added to the Excel formulation one constraint describing all the polygons and solved a problem with one objective and one constraint. The solution was found in a couple of seconds at (377,756) which is exactly in the city of Haifa. The maximum distance of 468 km is equidistant from Cairo and Adana.

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Figure 1: The Polygon

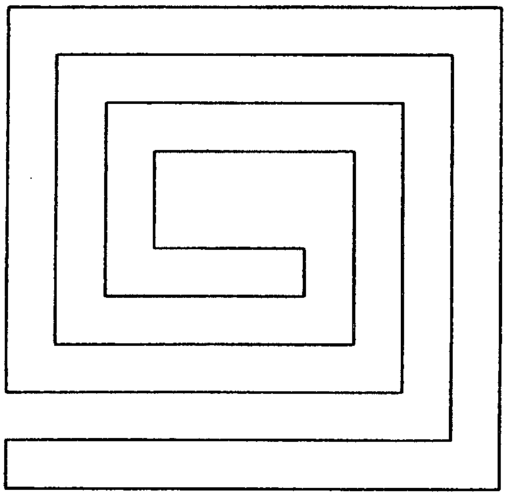


Figure 2: Random Points inside the Polygon

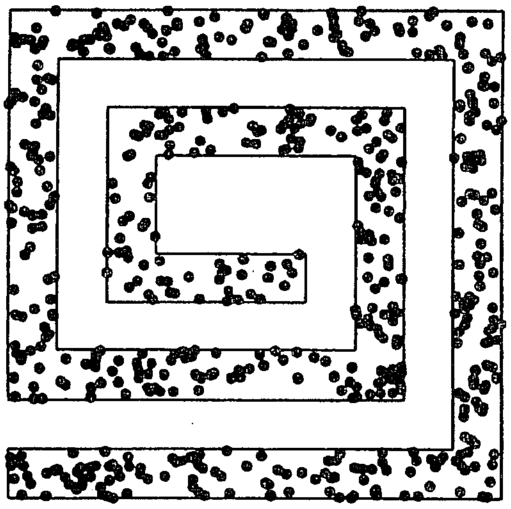
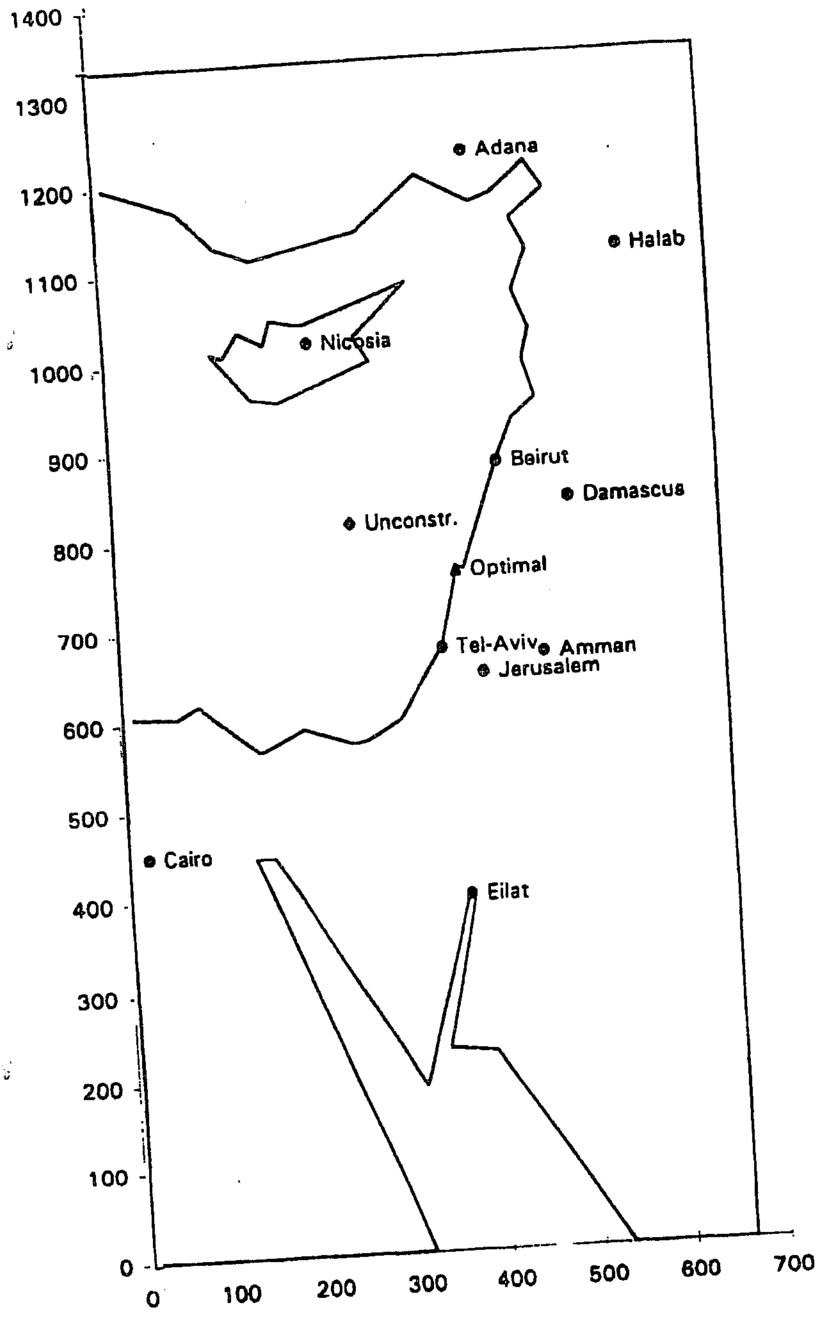


Figure 3.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