



Journal of Management Sciences and
Regional Development
Issue 2, January 1999
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<http://www.stt.aegean.gr/geopolab/GEOPOL%20PROFILE.htm>
ISSN 1107-9819
Editor-in-Chief: Abraham Mehrez

A SUPPLY CHAIN INVENTORY MODEL WITH REWORK COST AND VARIABLE UNIT PRODUCTION TIMES

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Abstract. The economic lot delivery problem is to determine the production and delivery interval of a component produced at a supplier site and shipped to an assembly facility which uses it at a constant rate. The objective is to find the production and delivery interval that minimizes the holding, setup, and transportation cost for the supply chain.

Previous solutions to the problem assume a constant production rate for each component at the supplier and that all the quantity produced is of acceptable quality. This assumption does not take into account volume flexibility and quality deterioration. Volume flexibility permits a manufacturing system to adjust production upwards or downwards within wide limits. Also, component quality may deteriorate with larger lot sizes and decreased unit production times. In this paper, we develop and solve an economic lot delivery model for a supplier using a volume flexible production system where component quality depends on both the lot size and unit production time. We illustrate the models with numerical examples.

Key words: Supply chain, inventory management, quality management.

The author would like to thank the referees for their helpful suggestions.

1. INTRODUCTION

Consider the problem faced by a supply chain consisting of a supplier that produces a component on a single production line or machine, accumulates the component, and then delivers it to an assembly facility [Hahm and Yano, 1992]. The assembly facility uses the component at a constant rate and may be located in a different country than the supplier facility as is the case of global supply chains [Flaherty, 1996]. Examples of such a supply chain are common in the automotive industry where suppliers include engine and transmission plants and bumper and fabrication assembly [Hahm and Yano, 1995a, 1995b]. The supplier incurs the following costs:

1. The setup cost of the production system. This is the cost of adjusting the machines and equipment before the production of the component [Hax and Candea, 1984].
2. The holding cost. This is the cost of holding inventory until it is shipped to the assembly facility. This cost includes the cost of capital tied up in inventory, storage, and insurance.

The assembly facility incurs only a holding cost for the component. This cost depends on the quantity shipped from the supplier at a time. The larger the quantity, the larger the average inventory and the annual holding cost. The economic lot delivery problem (ELDP) is to find the delivery time interval that would minimize the total costs of setup, holding, and shipping for the supply chain.

Hahm and Yano [1992] provided an excellent review of the literature relating to the ELDP and solved the problem for the general case of allowing M equally spaced deliveries to the assembly facility within a single supplier production cycle. Additionally, the authors used the model to provide some important insights about just-in-time inventory (JIT) management. Their model can be used to determine the reduction needed in setup cost and setup time in order to synchronize production and delivery (i.e. make $M=1$). Extensions of the ELDP deal with multiple components which is known as the economic lot and delivery scheduling problem (ELDSP). The ELDSP is to find the production sequence and delivery time interval that would minimize the total costs of setup, holding, and shipping for the supply chain. Hahm and Yano [1995a] provided an excellent review of the literature relating to the ELDSP, developed an efficient algorithm for solving it, and tested the performance of the

algorithm. Hahm and Yano [1995b, 1995c] extended the analysis to the case where equally spaced multiple shipments from the supplier to the assembly facility are allowed.

In the ELDP model developed by Hahm and Yano [1992], as with many other inventory models, unit production time (or alternatively production rate) is assumed to be constant and all units produced are assumed to be of perfect quality. However, volume flexibility of modern production systems and the quality management literature suggests otherwise.

Over the last two decades manufacturing flexibility has become much more important to firms and advances in computer technology made it less expensive to acquire. Among the most important types of flexibility is volume flexibility [Slack, 1987]. Volume flexibility of a manufacturing system is defined as its ability to be operated profitably at different overall output levels. Volume flexibility permits a manufacturing system to adjust production upwards or downwards within wide limits [Sethi and Sethi, 1990].

Stability of manufacturing costs over widely varying levels of production volume has been suggested as a measure of volume flexibility [Falkner, 1986]. Ramasesh and Jayakumar [1991] used the shape of the average unit cost curve over the range of production volumes as a measure of volume flexibility. The idea is that the more volume flexible a production system is, the flatter its U-shaped average unit cost curve is around the "design volume". The reason that the average unit cost function is decreasing until the "design volume" is reached is that fixed costs are spread over more units as production volume increases. After reaching the "design volume" increases in tool wearout and increased proportions of defectives [Schweitzer and Seidmann, 1991; Drozda and Wick, 1983] cause the average unit cost to increase.

This new capability should be included in scheduling and inventory models. Actually, volume flexibility of production systems has already led to the modification of inventory and scheduling models [Khouja, 1995; Khouja and Mehrez, 1994; Gallego, 1993; Moon, Gallego, and Simchi-Levi, 1991; Silver, 1990].

The assumption of perfect quality has also been found to be unrealistic in many cases. The earliest models assuming imperfect quality were proposed by Porteus [1986] and Rosenblatt and Lee [1986]. Porteus assumed the production process to be perfectly functioning at the start of production of a lot. With the production of each unit the process gradually shifts toward "out-of-control". In other words, the process follows a two-state Markov chain during the production of a lot, with transition occurring with each unit produced. Based on that assumption, the number of conforming units in a lot is a random variable that depends on the transition probabilities of the Markov chain and the production

lot size. Using the results from this assumption, the author derived the optimal lot size. Rosenblatt and Lee [1986] assumed the production process to be perfectly functioning at the start of the production of a lot. At some time, τ , the process shifts "out-of-control" and starts to produce α percent of defective products. Furthermore, τ is assumed to have a negative exponential distribution with mean $1/\mu$. The number of conforming units in a lot is thus a random variable that depends on α and μ . The authors then derived the optimal lot size. Cheng [1991] proposed an extension to the EOQ model in which demand exceeds supply and the production process is imperfect. Thus, the amount of demand the firm chooses to satisfy becomes a decision variable. Unit production cost was assumed to decrease as more demand is satisfied (economies of scale) and to increase as process reliability is improved (due to investment in technology). The author then derived closed form expressions for the optimal demand to be satisfied, the order quantity, and process reliability. Khouja and Mehrez [1994] formulated an economic production lot size (EPL) model that treats production rates as a decision variable and assumes the percentage of acceptable quality components in a lot to decrease with increased production rate. The authors used the quality deterioration assumptions used by Rosenblatt and Lee [1986] but were unable to provide closed-form solutions to their model.

The goal of this paper is to formulate and solve an ELDP model that takes into account the following:

1. Unit production time, or alternatively production rate, is a decision variable.
2. Quality of the production process deteriorates with decreased unit production time.
3. Unit production cost is a function of unit production time.

The above extension is important because the ELDP is common in managing supply chains. A supply chain is a group of organizations that transform raw materials into intermediate goods, then to final goods, and finally deliver them to customers [Flaherty, 1996]. Managing supply chain inventory has been identified as an important problem in operations management [Lee and Billington, 1992]. Actually, Lee and Billington identified the use of simplistic inventory stocking policies and the incorrect assessment of inventory costs which includes omitting rework cost as pitfalls of managing supply chain inventory. The inclusion of rework cost is of particular importance when the supply chain operates across borders and the incentive to increase the production lot size to reduce the setup and transportation costs is quite high. Examples of supply chains operating across borders are more common in regions with smaller countries such as South East Asia, the Middle East and Africa. It is

not uncommon that the suppliers for an assembly operation in one country are located in one or more neighboring countries.

The proposed model shows that the inclusion of rework cost can lead to large reduction in the production lot size and the rework cost.

In section 2, we formulate the ELDP under the classical assumptions, identify the optimal solution, and then reformulate the ELDP with unit production times treated as decision variables and imperfect quality and provide closed-form optimal solution. In section 3, we generalize the problem to the case where unit production cost is a function of unit production time and identify the optimal solution for a realistic cost function. Numerical examples are presented in section 4. Section 5 contains the conclusion and suggestions for future research.

2. BASIC MODELS AND THEIR SOLUTION

Notation

- D = demand per unit time,
- S = setup cost,
- s = setup time,
- H = inventory holding cost (\$ per unit per unit time),
- A = transportation cost per delivery (\$ per shipment),
- R = rework cost (\$ per unit),
- Q = the lot size, a decision variable,
- T = setup interval (time between setups); also equal to the delivery interval (time between deliveries), $Q = TD$,
- p = unit production time.

The total cost for the supply chain per unit time is identical to the cost in the EOQ with the addition of shipment cost and the cost of accumulating the inventory at the supplier site given by $(Q/2)QpH$ per cycle, or $(Q/2)QpH(D/Q) = (Q/2)DpH$ per unit time. Thus

$$TC = \frac{D}{Q}(S + A) + \frac{Q}{2}H + \frac{Q}{2}DpH, \quad (1)$$

A feasible solution must have a delivery cycle T long enough to accommodate the setup time s and the production time Qp . Thus, $s + Qp \leq T$. Using $Q = DT$ gives:

$$s + DTp \leq T \text{ or } Ds + DQp \leq Q \quad (2)$$

Define

$$\tau = s / (1 - pD) \text{ and } \omega = Ds / (1 - Dp) \quad (3)$$

Setting the first derivative of TC' to zero as in the EOQ gives the unconstrained optimal lot size:

$$Q_1^* = \sqrt{\frac{2D(S+A)}{H(1+Dp)}} \quad (4)$$

Since $Q = DT$, the optimal delivery interval is

$$T_1^* = \sqrt{\frac{2(S+A)}{DH(1+Dp)}} \quad (5)$$

Note that holding inventory at the supplier site acts to decrease the optimal lot size as compared to the EOQ. Since constraint (2) may be binding, the optimal solution is may be given by (3). The optimal solution is in general given by:

$$Q^* = \max\{Q_1^*, \omega\}, \text{ or } T^* = \max\{T_1^*, \tau\} \quad (6)$$

Now suppose that the production process is imperfect. Similar to assumptions made by Porteus [1986], we assume the process to be perfectly functioning at the start of production of a lot. With the production of each unit the process gradually shifts toward "out-of-control" and starts producing defectives, with probability q . Once the process shifts to the out control state, it stays that way until the remainder of the lot is produced. In other words, the process follows a two-state Markov chain during the production of a lot, with transition occurring with each unit produced and transition probability q . Porteus gave three interpretation for the above assumption on process quality. The first interpretation is that the firm inspects only the first and last pieces as suggested by Hall [1983] and if the last piece is

good then the entire lot is judged to be good. The second interpretation is that the type of defect cannot be identified until the product is processed on the next station and the entire lot is produced before it is moved to the next station. The third interpretation is that even though the process continuously monitored, there is a delay between the time the process shifts out of control and the time the operators can determine the process is out of control. The monitoring is still useful in determining which pieces need rework.

Based on this assumption, for small values of q , Porteus shows that the expected number of defectives per lot of size Q is $E(U) = Q - g(1 - g^Q)/q$ where $g = 1 - q$. Porteus showed that $E(U)$ is a strictly increasing, strictly convex function of Q . For example, for a small transition probability of $q = 0.002$ and a lot of size $Q=100$, the expected number of defectives is $E(U) = 9.5$. For a lot size of $Q=200$, the expected number of defectives increases to $E(U) = 35.4$. The expected proportion of defectives is given by $E(U)/Q = (Q - g(1 - g^Q)/q)/Q$ and the rework cost per unit time is $DR[(Q - g(1 - g^Q)/q)/Q]$. The total cost per time unit, including the rework cost, is

$$TC = \frac{D}{Q}(S + A) + \frac{Q}{2}H + \frac{Q}{2}DpH + DR - \frac{DRg(1 - g^Q)}{qQ}. \quad (7)$$

If q is close to zero, then Portues [1986] shows that $(Q - g(1 - g^Q)/q)/Q \equiv \frac{1}{2}qQ$. The rework cost is thus approximately $(Q/2)DRq$ which when used in (7) gives total cost per unit time:

$$TC = \frac{D}{Q}(S + A) + \frac{Q}{2}H + \frac{Q}{2}DpH + \frac{Q}{2}DRq. \quad (8)$$

Case 1. In this case q is a constant that is determined by the manufacturing technology as assumed by Portues [1986]. Setting the first derivative of TC in (8) to zero gives the optimal lot size:

$$Q_2^* = \sqrt{\frac{2D(S + A)}{H(1 + Dp) + DRq}}, \quad (9)$$

and the optimal delivery interval:

$$T_2^* = \sqrt{\frac{2(S + A)}{D[H(1 + Dp) + DRq]}}. \quad (10)$$

Since (9) and (10) may not always yield a feasible solution, the optimal solution is given by:

$$Q^* = \max\{Q_z^*, \omega\}, \text{ or } T^* = \max\{T_z^*, \tau\}, \quad (11)$$

where ω and τ are given by (3).

Case 2. If the production system is volume flexible, then p is a decision variable. Also, the transition probability q is expected to be a function of p . The smaller the unit production time (higher production rate) the faster the deterioration in quality and the larger the q . Thus, $q = f(p)$ where f is a decreasing function.

Suppose $q = \alpha / p$ which implies that the quality of the process deteriorates at an increasing rate with decreased unit production times. Substituting in (8) gives the minimization problem:

$$\text{Minimize } TC = \frac{D}{Q}(S+A) + \frac{Q}{2}H + \frac{Q}{2}DpH + \frac{Q}{2}DR\frac{\alpha}{p} \quad (12)$$

$$\text{subject to } Ds + DQp - Q \leq 0 \quad (13)$$

The necessary conditions for optimality [Bazaraa and Shetty, 1979] are:

$$-\frac{D}{Q^2}(S+A) + \frac{H}{2} + \frac{DpH}{2} + \frac{DR\alpha}{2p} + \lambda(Dp-1) = 0 \quad (14)$$

$$\frac{Q}{2}DH - \frac{1}{p^2} \frac{Q}{2}DR\alpha + \lambda DQ = 0 \quad (15)$$

$$\lambda(Ds + DQp - Q) = 0 \quad (16)$$

If constraint (13) is not binding then $\lambda = 0$ and from (15)

$$p = \sqrt{\frac{R\alpha}{H}} \quad (17)$$

Equation (17) suggests that the larger the unit rework cost, the slower the production rate (larger p) since slowing down production will avoid increases in the number of components that need rework. Also, the larger the holding cost the faster the production rate (smaller p) since speeding up production will avoid having many components waiting for a long time at the supplier site to be shipped to the assembly facility. From (14)

$$Q_s^* = \sqrt{\frac{2Dp(S+A)}{Hp(1+Dp)+DR\alpha}} \quad (18)$$

Using $Q = DT$ in (18), the optimal delivery interval is

$$T_s^* = \sqrt{\frac{2p(S+A)}{D(Hp(1+Dp)+DR\alpha)}} \quad (19)$$

If the constraint is binding then $\lambda > 0$ and from (16)

$$Q = \frac{Ds}{(1-Dp)} \text{ or } T = \frac{s}{(1-Dp)} \quad (20)$$

Eliminating λ from (13) and (15) gives

$$g(p) = 2p^2 \left[H - \frac{(1-Dp)^2}{Ds^2} (S+A) \right] + \alpha R(2Dp-1) = 0 \quad (21)$$

$g(p)$ is a fourth degree polynomial that may have up to 4 roots. Lemma 1 proves the existence of at least one feasible root and at most three feasible roots that satisfy the Kuhn-Tucker necessary conditions. Lemma 2 shows that when $(S+A) > 4DHS^2$, the optimal unit production time (p^*) can be found using a simple numerical search on the interval $[1/2D, 1/D)$. The condition $(S+A) > 4DHS^2$ is satisfied for any reasonable real world problem since s is the setup time and s^2 is very small compared to other parameters. For example, for setup cost of $S = \$100$ and shipment cost of $A = \$200$, $(S+A) = \$300$, and for demand of $D = 100,000$ units/year, $H = \$5/\text{unit-year}$, and $s = 2$

hours, $4DHS^2 = 2$ (assuming 250 days per year and 8 hours per day) and the inequality easily holds. Even if demand is increased to $D = 1000,000$ and holding cost is increased to $H = \$10$, $4DHS^2 = 40 < 300$. Once p^* is found using a method such as the Bisection [Maron, 1982], it is used in (20) to compute the optimal order quantity.

Lemma 1. *There is at least one feasible solution and at most three feasible solutions that satisfy the Kuhn-Tucker necessary conditions (14-16).*

Proof. The proof of Lemma 1 is provided in the Appendix.

Lemma 2. *If $(S + A) > 4DHS^2$ then the optimal unit production time is given by the single root of $g(p)$ on the interval $[1/2D, 1/D)$.*

Proof. The proof of Lemma 2 is provided in the Appendix.

The Quality of the Approximation of Rework Cost and the Value of q .

The above derivations are based on the following approximation of the expected proportion of defectives in a lot of size Q developed by Porteus [1986] and valid for small values of q :

$$1 - \frac{g(1 - g^Q)}{qQ} \cong \frac{1}{2} qQ. \quad (22)$$

The error in the approximation of the proportion of defectives is calculated by subtracting the right hand side of (22) from the left hand side and dividing by the right hand side which gives:

$$\text{Percentage error in proportion of defectives} = \left[1 - \frac{q^2 Q^2}{2[qQ - g(1 - g^Q)]} \right] \times 100. \quad (23)$$

As Figure 1 shows, the approximation is quite accurate (within 5% of the actual) for values to q up to 0.0012 for $Q=100$. When the approximation in (22) results in a large percentage error, which is computed using (23), for given values of Q and q , a correction can be incorporated into the model. First, an initial Q (and q if case 2 is applicable) is computed according to the formulas given for cases 1 or 2. This gives the approximate range of the optimal Q and q . If (23) shows that (22) is resulting in a

poor approximation of the proportion of defectives for the initial values Q and q , then a numerical correction factor C which brings the values of the two sides of (22) closer to each other for the initially computed values of Q and q can be applied. The correction factor C is calculated by setting the percentage error in (23) to zero and multiplying the negative term of the right hand side by C and solving for C which gives:

$$C = 2 \left[1 - \frac{g(1-g^2)}{qQ} \right] / [qQ]. \quad (24)$$

The correction factor C is used in (8) to improve the approximation of the proportion of defectives and rework cost at the optimal values of Q and q resulting in:

$$TC = \frac{D}{Q}(S+A) + \frac{Q}{2}H + \frac{Q}{2}DpH + \frac{Q}{2}CDRq. \quad (25)$$

The optimal order quantity in (9) becomes:

$$Q_2^* = \sqrt{\frac{2D(S+A)}{H(1+Dp)+CDRq}}. \quad (26)$$

Similar corrections can be made in (17), (18), and (21) by multiplying α by C .

Porteus [1986] did not use a correction factor since for realistic values of Q and q the error of the approximation is small. Because the error of the approximation is small, it is sufficient to compute the correction factor, if needed, only once and not until convergence. In other words, the correction factor is computed only for the initial values of Q and q , and using that constant Q and q are recomputed. The use of the correction factor will be demonstrated using a numerical example.

3. COMPONENT QUALITY AND COSTS AS FUNCTIONS OF PRODUCTION TIME

In this case, in addition to component quality, component production cost is a function of its unit production time. Thus, $q = \alpha / p$ and component production cost is $h(p)$, where

$h(p)$ = average unit production cost as a function of unit production time. We assume $h(p)$ to be a convex function with a minimum at p_m which can be considered the “design unit production time” [Ramasesh and Jayakumar, 1991; Sethi and Sethi, 1990].

Consider the following unit production cost function:

$$h(p) = r + ap + b / p, \quad (27)$$

where

r , a , and b are non-negative real numbers to be chosen to provide the best fit for the estimated unit production cost function. For the above function:

r is a cost component independent of unit production time and includes raw material cost,

ap is a per unit cost component that decreases with decreased unit production time. This cost component would include labor cost. For example, if one worker is needed to tend the machine, then as more components are produced per unit time, the wages of the worker are spread over more units. In other words, a is what the literature on optimizing machining rates refers to as cost of operating time (\$/unit time) [Petrooulos, 1973].

b / p is a cost component that increases with decreased production time per unit and includes tool cost that might result from increased tool wearout at higher production rates.

The unit production time that minimizes unit production cost is

$$p_m = \sqrt{\frac{b}{a}}, \quad (28)$$

and the minimum unit cost is

$$C_m = h(p_m) = r + 2\sqrt{ab}. \quad (29)$$

Adding the production cost to (12) gives the following minimization problem:

$$\text{Minimize } TC = \frac{D}{Q}(S + A) + \frac{Q}{2}H + \frac{Q}{2}DpH + \frac{Q}{2}DR\frac{\alpha}{p} + D(r + ap + b / p) \quad (30)$$

Note that the holding cost was treated as independent of unit cost. This is an approximation which simplifies the analysis and is reasonable because 1) the cost of capital is only one component of the inventory holding cost, and 2) volume flexibility leads to small changes in unit production cost as a result of changes in unit production time. For example, suppose the holding cost is charged at 15% percent for a product that is \$40 per unit. If the cost of capital is 8% and the change in unit production time leads to a 5% change in unit cost, then this approximation will result in understating unit holding cost by 3% (\$6 instead of \$6.16 per unit per year) which will have a very small effect when viewed relative to the total cost. At the end of this section we provide derivations for the case where exact results are desired. The necessary optimality conditions [Bazaraa and Shetty, 1979] are the same as (14-16) except for (15) which becomes:

$$\frac{Q}{2}DH - \frac{1}{p^2} \frac{Q}{2}DR\alpha + Da - \frac{Db}{p^2} + \lambda DQ = 0 \quad (31)$$

If the constraint is not binding then $\lambda = 0$ and from (31)

$$p = \sqrt{\frac{2b + QR\alpha}{QH + 2a}} \quad (32)$$

Lemmas 3 and 4 show that only one solution satisfies the Kuhn-Tucker necessary conditions and that (18) and (32) will converge to that solution. Lemma 3 applies when $\sqrt{R\alpha/H} > \sqrt{b/a} = p_m$. In this case, the expected rework cost at p_m is greater than the holding cost. Lemma 3 shows that the optimal unit production time is greater than p_m which reduces the proportion of defectives and the rework cost. Lemma 4 applies when $\sqrt{R\alpha/H} < \sqrt{b/a} = p_m$. In this case, the expected rework cost at p_m is smaller than the holding cost. Lemma 4 shows that the optimal unit production time is smaller than p_m . Lemma 4 holds when $\partial^2 Q / \partial p^2$ evaluated at p_m is positive. Numerical analysis shows that this condition is satisfied for any realistic problem parameters. If the condition does not hold, more than one solution that satisfy the Kuhn-Tucker conditions are possible.

Lemma 3. *If the capacity constraint is not binding and $\sqrt{R\alpha/H} > \sqrt{b/a}$, then $p^* > p_m$ and by initiating equation (18) with $p = p_m$, equations (18) and (32) will converge to the only solution that satisfies the Kuhn-Tucker necessary conditions.*

Proof. The proof of Lemma 3 is provided in the Appendix.

Lemma 4. *If the capacity constraint is not binding, $\sqrt{R\alpha/H} < \sqrt{b/a}$, and $\frac{\partial^2 Q}{\partial p^2}(p = \sqrt{b/a}) < 0$, then*

$p^ < p_m$ and by initiating equation (18) with $p = p_m$, equations (18) and (32) will converge to the only solution that satisfies the Kuhn-Tucker necessary conditions.*

Proof. The proof of Lemma 4 is provided in the Appendix.

If the capacity constraint is binding then eliminating λ from (14) and (31) gives

$$c_1 p^4 + c_2 p^3 + c_3 p^2 + c_4 p + c_5 = 0, \quad (33)$$

where

$$c_1 = 2D[a - (A + S)/s],$$

$$c_2 = 4(A + S)/s + DIIs - 2Ds\left(\frac{H}{2} + \frac{a'}{Ds}\right) - 2a,$$

$$c_3 = IIs + 2s\left(\frac{H}{2} + \frac{a}{Ds}\right) - 2(A + S)/Ds - 2Db,$$

$$c_4 = DRs\alpha + 2\left(\frac{\alpha R}{2} + \frac{b}{Ds}\right)Ds + 2b, \text{ and}$$

$$c_5 = -2s\left(\frac{\alpha R}{2} + \frac{b}{Ds}\right).$$

Since the optimal solution when constraint (2) is binding is such that $1 - pD \approx 0$, a search for the root of (33) on the range $(1/2I, 1/D)$ will identify the optimal unit production time.

Unit Holding Cost is a Fraction of Unit Cost

In this case, the holding cost is given by $H = I \times h(p)$ where I is the fraction holding cost which depends on the cost of capital. Substituting for H in equation (30) and solving the Kuhn-Tucker necessary conditions for the case when the constraint is not binding gives:

$$Q_3^* = \sqrt{\frac{2Ip(S + A)}{Ip(r + ap + b/p)(1 + Dp) + DR\alpha}}, \quad (34)$$

and

$$B_4 Q p^3 + (Da + QIB_2/2)p^2 - (QB_3/2 - Db) = 0. \quad (35)$$

where

$$B_1 = I(r + Db),$$

$$B_2 = a + rD,$$

$$B_3 = Ib + DR\alpha,$$

$$B_4 = IDa.$$

Similar to solving (18) and (32), equations (34) and (35) are solved in an iterative fashion with $p = p_m$ providing the initial value in (34). When the capacity constraint is binding, solving the Kuhn-Tucker conditions requires solving a fifth degree polynomial in p . In this case the approximation of (18) and (33) can be used to obtain a good solution. Actually, as the numerical example will show, the use of a fixed holding cost provides a good approximation to the case where holding cost is treated as a fraction of unit cost.

4. NUMERICAL EXAMPLES

Consider a supply chain where the assembly facility has a demand of $D=2000$ units per year, setup cost is $S=\$100$, setup time is $s=2$ hours (0.001 years), shipment cost is $A=\$100$, holding cost is $H=\$8$ per unit per year, rework cost is $R=\$25$ per unit, unit production time is $p=0.00025$ years/unit (0.5 hours/unit), and transition probability is $q=0.0004$. The supply chain operates 250 days per year and 8 hours per day. Ignoring component quality cost and applying (6) gives $Q^*=258$ and a total cost (including cost of rework) of $TC=\$5678$ per year.

Case 1: Quality is independent of unit production time. When quality cost is taken into account, (11) gives $Q^*=158$ and the total cost drops to $TC=\$5060$, a saving of 10.9%.

Case 2: Quality is a function of unit production time. In this case, quality deteriorates with decreased unit production time. Suppose $q = 1 \times 10^{-7} / p$ and demand is $D=1000$ units per year. Using (17) and (18) gives an optimal unit production time of $p^*=0.00055902$ years per unit (1.118 hours per

unit) which results in a transition probability of $q^*=0.00017889$, an optimal lot size of $Q^*=153$, and a total cost of $TC=\$2603$ per year. If demand is increased to $D=2000$ units per year, the capacity constraint becomes binding and solving (21) gives an optimal unit production time of $p^*=0.0004943$ years per unit (0.989 hours per unit) which results in a transition probability of $q^*=0.0002023$, an optimal lot size of $Q^*=175$, and $TC=\$4563$ per year.

The Quality of the Approximation of Rework Cost and the Value of q . To illustrate the use of the correction factor, consider the above example with $q = 4 \times 10^{-6} / p$, $D=1000$, $H=\$15$, and $R=\$3$. Using (17) and (18) gives an optimal unit production time of $p^*=0.000894$ years per unit which results in a transition probability of $q^*=0.00447$ and an optimal lot size of $Q^*=97$. Using (23) shows that for $q^*=0.00447$ and $Q^*=97$, the approximated percentage of defectives is above the actual by 13.7%. Using (24) gives a correction factor of $C=0.8799$ which when used in (17) and (18) gives $Q^*=99$ and $p^*=0.000839$ which gives $q^*=0.00477$. With the correction factor, the approximated percentage of defectives is above the actual by only 3%. The use of the correction factor causes the actual annual cost to drop from $\$4,111$ to $\$4,094$.

Case 3: Quality and production cost are functions of unit production time. Suppose $q = 3 \times 10^{-7} / p$ and $h(p) = 24 + 32000p + 0.002 / p$ with a minimum unit cost of $C_m=\$40$ at $p_m=0.00025$ years per unit (0.5 hours per unit). To illustrate the effect of having the supplier and assembly facility be in different countries, we use a shipment cost of $A=\$1000$ and assume that defective units are reworked at a high cost of $R=\$50$ per unit to avoid a shortage at the assembly facility. For a demand of $D=2000$ per year, ignoring volume flexibility and leaving unit production time at $p_m=0.00025$, (9) gives $Q^*=182$ and the total cost (including production cost) is $TC=\$104,100$. When volume flexibility is taken into account, (18) and (32) converge to $p^*=0.0003280$ years per unit and $Q^*=204$ with a total cost of $TC=\$102,163$ per year. For a demand of $D=3500$, ignoring volume flexibility and leaving unit production time at $p_m=0.00025$, (9) gives $Q^*=184$ and total cost $TC=\$181,624$. When volume flexibility is taken into account, (18) and (32) converge to an infeasible solution. Solving (33) gives $p^*=0.000281$ for which (20) gives $Q^*=212$ and total cost $TC=\$180,030$ per year.

Unit Holding Cost is a Fraction of Unit Cost. Reconsidering the above example with $D=2000$ and using (34) and (35) in which the holding cost is treated as a fraction of unit cost gives $p^* = 0.0003286$ years per unit, $Q^* = 210$, and $TC=102,140$ (instead of $p^* = 0.0003280$, $Q^* = 204$, and $TC=\$102,163$) resulting in a small saving of \$23 per year or 0.023%.

5. CONCLUSION AND SUGGESTION FOR FUTURE RESEARCH

In this paper, we developed and solved an economic lot delivery model for a supplier using a volume flexible production system where component quality depends on both the lot size and unit production time. The results show that incorporating quality cost results in producing at unit production time different than the one that minimizes average unit production cost. The results also show that the larger the rework cost, the larger the unit production time. Also, the larger the holding cost, the smaller the unit production time. The deviation of the optimal unit production time from the production time that minimizes the average unit production cost is influenced by flexibility of the system with smaller deviations associated with inflexible systems.

Future extensions to the above problem includes allowing multiple equally spaced shipments from the supplier to the assembly facility within one supplier production cycles [Hahm and Yano, 1992]. Other research includes considering volume flexibility and quality cost in the multiple components problem which is known as the economic lot and delivery scheduling problem (ELDSP).

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Appendix

Proof of Lemma 1. Since $g(p)$ is a fourth degree polynomial, it can have at most four real roots. For p to be feasible, (2) requires $1 - pD > 0$ which gives $p < 1/D$. Evaluating $g(p)$ at $p = 0$ and $p = 1/D$ gives $g(0) = -R\alpha$ and $g(1/D) = 2H/D^2 + R\alpha$. Since $g(0) < 0$ and $g(1/D) > 0$ there is at least one root in the feasible interval $(0, 1/D)$.

At p sufficiently large, the sign of $g(p)$ is determined by the fourth power term which is negative for $pD < 1$. Since $g(1/D) > 0$ and for p sufficiently large $g(p) < 0$ there is at least one root on the infeasible range $(1/D, \infty)$. Since $g(p)$ can have at most four roots and since there is at least one root in the infeasible range, $g(p)$ can have at most three feasible roots.

Proof of Lemma 2. Rewrite $g(p)$ as

$$g(p) = 2p^2H + R\alpha(2Dp - 1) - 2p^2 \frac{(1 - Dp)^2}{Ds^2} (S + A). \quad (\text{A.1})$$

Define

$$h(p) = -2p^2 \frac{(1 - Dp)^2}{Ds^2} (S + A). \quad (\text{A.2})$$

Taking the first derivative

$$h'(p) = -4p(1 - 3Dp + 2D^2p^2), \quad (\text{A.3})$$

with zeros at $p = 0, 1/2D$, and $1/D$. Testing the first derivative shows that $h(p)$ is decreasing on $(0, 1/2D)$, increasing on $(1/2D, 1/D)$, and decreasing on $(1/D, +\infty)$. Since the other two terms of $g(p)$ are increasing, $g(p)$ is increasing on $(1/2D, 1/D)$. Evaluating $g(p)$ at $p = 1/2D$ yields:

$$g(1/2D) = \frac{1}{2D^2} \left[H - \frac{S + A}{4Ds^2} \right]. \quad (\text{A.4})$$

Since $(S + A) > 4DHS^2$, $g(1/2D) \leq 0$. Thus, $g(p)$ is increasing on $(1/2D, 1/D)$, $g(1/2D) \leq 0$, and from the proof of Lemma 1, $g(1/D) > 0$ which implies that there is a single root of $g(p)$ on

$(1/2D, 1/D)$. Since the optimal solution when constraint (2) is binding is such that $1 - p/D \approx 0$ the root on $(1/2D, 1/D)$ is optimal.

Proof of Lemma 3. Taking the first derivative of Q in (18) with respect to p gives:

$$\frac{\partial Q}{\partial p} = (R\alpha - p^2 H) \sqrt{\frac{D^3(S+A)}{2p(Hp + HDp^2 + DR\alpha)^3}}. \quad (A.5)$$

Taking the second derivative,

$$\frac{\partial^2 Q}{\partial p^2} = -\sqrt{\frac{2D^3(S+A)}{(Hp + HDp^2 + DR\alpha)^3}} \left[\sqrt{pH^2} + \sqrt{\frac{4Hp + 7HDp^2 + DR\alpha}{8p^3(Hp + HDp^2 + DR\alpha)^2}} (R\alpha - p^2 H) \right]. \quad (A.6)$$

Taking the first and second derivatives of p in (32) with respect to Q gives:

$$\frac{\partial p}{\partial Q} = (aR\alpha - bH) \sqrt{\frac{1}{(2b + QR\alpha)(QH + 2a)^3}}, \quad (A.7)$$

and

$$\frac{\partial^2 p}{\partial Q^2} = -(aR\alpha - bH)(2QRH\alpha + Ra\alpha + 3Hb) \sqrt{\frac{1}{(2b + QR\alpha)(QH + 2a)^3}}. \quad (A.8)$$

If $\sqrt{\frac{R\alpha}{H}} > \sqrt{\frac{b}{a}}$, then $aR\alpha > bH$, $\frac{\partial p}{\partial Q} > 0$ and $\frac{\partial^2 p}{\partial Q^2} < 0$ and p is increasing and concave which when plotted with p on the horizontal axis results in the shape shown in Figures 2, 3, and 4. From (A.5), Q is increasing for $p < p_c = \sqrt{R\alpha/H}$ and from (A.6) Q is concave for at least the same range. Let p_l be the point where Q changes from concave to convex (inflection point). As Figures 2, 3, and 4 show three cases are possible:

1. $p_m < p^* < p_c < p_l$ shown in Figure 2,
2. $p_m < p_c < p^* < p_l$ shown in Figure 3, and
3. $p_m < p_c < p_l < p^*$ shown in Figure 4.

As the figures show, in all three cases there is only one solution that satisfies the Kuhn-Tucker conditions and by initiating (18) with $p = p_m$, (18) and (32) will converge to that solution.

Proof of Lemma 4.

If $\sqrt{\frac{R\alpha}{H}} < \sqrt{\frac{b}{a}}$, then $aR\alpha < bH$, $\frac{\partial p}{\partial Q} < 0$ and $\frac{\partial^2 p}{\partial Q^2} > 0$ and p is decreasing and convex which when plotted with p on the horizontal axis results in the shape shown in Figures 5 and 6. From (A.6), if $\frac{\partial^2 Q}{\partial p^2}(p = \sqrt{b/a}) < 0$ then one of two cases may arise as shown in Figures 5 and 6:

1. $\dot{p}^* < p_c < p_m < p_l$ shown in Figure 5, and
2. $p_c < p^* < p_m < p_l$ shown in Figure 6.

As the figures show, in both cases there is only one solution that satisfies the Kuhn-Tucker conditions and by initiating (18) with $p = p_m$, (18) and (32) will converge to that solution.