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GRAPH THEORETICAL MODELS FOR FREQUENCY ASSIGNMENT PROBLEMS : A HEURISTIC ALGORITHM FOR THE T-COLORING OF AN ARBITRARY GRAPH

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Abstract. The growth of the telecommunication technology led to the increment of transmission stations and therefore to the necessity for avoiding possible interference between the corresponding frequencies.

Graph theory is a convenient mathematical tool so to model a frequency assignment problem to known locations in order to confront interference situations.

Here we present the graph theory concepts of radiocoloring and T-coloring. We introduce some new invariants related to these notions that can be applied effectively to modelize and to subsequently solve variations of frequency assignment problems.

Finally an approximation algorithm for the T-chromatic number and for the T-chromatic cost is developed and a detailed numerical example of the proposed algorithm for a small graph is presented.

1. INTRODUCTION

The factor of telecommunication plays an important role to the regional development planning, since it facilitates the information interchange between peripheral and median districts. The information connection contributes considerably to the economical and cultural progression for the population of isolated areas and provides a powerful item of entertainment in benefit of their psychological disposition.

The quality of the transmissions (mobile telephone, radio, television, etc.) is significantly rectified by excluding interference between broadcast stations.

Frequency interference may occur between a subset of couples of stations for reason of closeness, geographic structure, atmospheric configuration, etc. Therefore the problem that arises refers to the assignment of frequencies to different stations taking into account interference occurrences. There exist certain variations of frequency assignment problems, depending on the number of channels allocated and on the sort of the required frequencies assigned at every station. For every frequency assignment problem a corresponding optimization question follows which depends on the considered measure, for example, the fewest number of used frequencies and the smallest deviation among the used frequencies.

The concept of graph coloring and some of its generalizations are suitable tools for describing visually diverse frequency assignment problems in order to solve them.

For reason of self-reliancy the next section is devoted to the notions of graph theory, see [1], [2] that are employed in the sequel. In Section 3 we present the notions of radiocoloring, T – coloring, see [3], [4] and we introduce new invariants which are generalizations of graph coloring. Section 4 is dedicated to the development of a ***heuristic algorithm*** see [5] that finds a good approximation for the T – chromatic number and T – chromatic cost for a given arbitrary graph. Also a detailed example of the working process of the algorithm is presented. The conclusions are the content of the last section.

1. DEFINITIONS – NOTATIONS

Let $V = \{v_1, v_2, \ldots, v_n\}$ be a nonempty set and $E \subseteq V \times V$ a subset of unordered couples (v_i, v_j), $v_i, v_j \in V$. The ordered pair (V, E) defines a graph $G = (V, E)$. The elements of V are usually called vertices, nodes or points and the elements of E links, lines or edges respectively. A graph can easily be drawn in the plane providing a good image of the connection structure of the elements of V for relatively small graphs.

Two nodes v_i and v_j are adjacent if they define an edge, i.e., $(v_i, v_j) \in E$. The set of adjacent nodes of a node $v_i \in V$ is denoted by $\Gamma(v_i)$, i.e., $\Gamma(v_i) = \{\, y$ such that $(v_i, y) \in E \,\}$ while the nonadjacent nodes of v_i are symbolized by

$\overline{\Gamma}(v_i) = V - \Gamma(v_i)$. The degree $\deg(v_i)$ of a node $v_i \in V$ expresses the number of adjacent nodes to v_i, obviously $\deg(v_i) = |\Gamma(v_i)|$.

The set of all adjacent nodes of all nodes in a set $S \subset V$ in the set $\Gamma(S) = \bigcup_{v_i \in S} \Gamma(v_i)$.

A path is a sequence $p = \{v_{i_1}, v_{i_2}, \dots, v_{i_k}\}$ of consecutive edges $(v_{i_{j-1}}, v_{i_j})$ such that $v_{i_j} \in \Gamma(v_{i_{j-1}})$ for $j = 2, 3, \dots, k$.

The distance $d(x, y)$ between two nodes $x, y \in V$ is the minimum number of edges that joins x and y with a path.

A graph $G' = (V', E')$ is a subgraph of $G = (V, E)$ if $V' \subset V$ and E' comprises only the edge of E that are produced by nodes in V', i.e., $E' = \{ (x,y) \in E \text{ and } x, y \in V' \}$. We also denote the subgraph G' of G by $G(V')$. The adjacent and nonadjacent nodes of $v \in V'$ in subgraph $G(V')$ are expressed by $\Gamma_{V'}(v)$ and $\overline{\Gamma}_{V'}(v)$ respectively.

A subset $S \subset V$ is called *independent set* in $G = (V, E)$ if for every pair $\{v_i, v_j\} \subset S$, v_i and v_j are not adjacent i.e., $v_i \notin \Gamma(v_j) \Leftrightarrow v_j \notin \Gamma(v_i)$. An independent set is *maximal* if it is not contained in any other independent set.

An assignment of colors to the nodes of a graph G so that adjacent nodes has different colors is a coloring of G. A coloring of G with n colors is a $n-$coloring and G is $n-$colorable. The minimum number of colors needed to perform a coloring of G is called the chromatic number $\gamma(G)$ of G. Obviously if $n = |V|$ and $\gamma(G) \le q \le n$ then G is $q-$colorable. In a coloring, the nodes that are assigned the same color form a coloring class. Clearly a coloring class $C_i \subseteq V$ is an independent set.

We define a reciprocal correspondence between the set of colors and the set of positive integers $I^+ = \{1, 2, 3, \dots\}$. In the continuation the colors will be referred by numbers in I^+. A graph coloring is a positive function c from the nodes of V to I^+, i.e., $c: V \to I^+$, where $c(y)$ express the color assigned to node $y \in V$. In the subsequent sections a color represents a specific frequency and inversely.

The above conceptions are illuminated in the subsequent figures for graph $G = (V, E)$, where $V = \{v_1, v_2, \dots, v_6\}$, $E = \{(v_1, v_2), (v_1, v_3), \dots, (v_5, v_6)\}$ and the colors in Figures 2a, 2b are represented by numbers of I^+.

$\Gamma(v_1) = \{v_2, v_3\}$, $\deg(v_1) = 2$

$\Gamma(v_2) = \{v_1, v_4, v_6\}$, $\deg(v_2) = 3$

....

$\Gamma(v_6) = \{v_2, v_4, v_5\}$, $\deg(v_6) = 3$

for $S = \{v_1, v_2\}$

$\Gamma(S) = \{v_1, v_2, v_3, v_4, v_6\}$

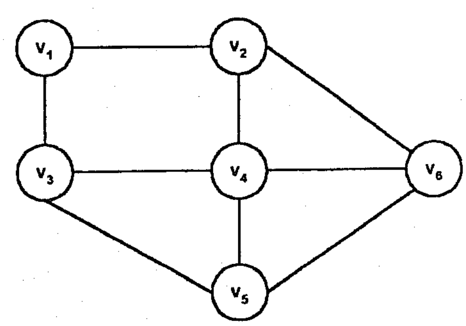

Figure 1. *Graph G*

$C_1 = \{v_1, v_4\}$
$C_2 = \{v_2, v_3\}$
$C_3 = \{v_5\}$
$C_4 = \{v_6\}$

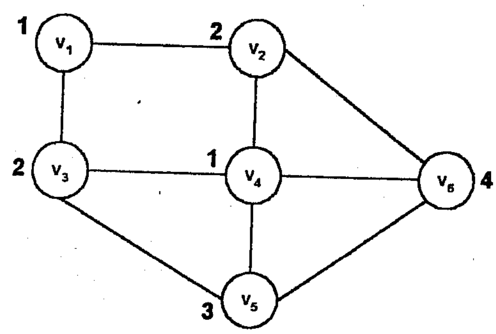

Figure 2-a. *4-coloring of G*

$C_1 = \{v_2, v_5\}$
$C_2 = \{v_1, v_4\}$
$C_3 = \{v_3, v_6\}$

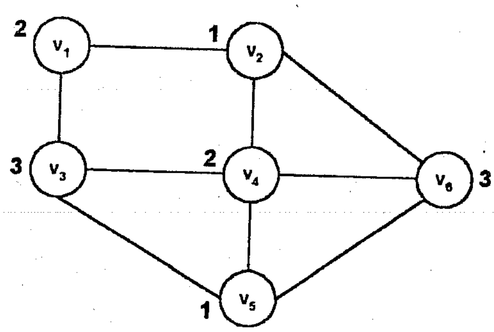

Figure 2-b. *3-coloring of G*

$G' = (V', E')$
$V' = \{v_2, v_3, v_4, v_6\}$

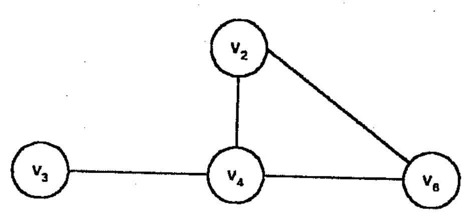

Figure 3. *Subgraph G' of G*

2. COLORING GENERALIZATION

In this section we present the generalization of the graph coloring concepts which are prominent for the modelization of frequency assignment problems. We also define invariants for every coloring type, that are suitable for the study of such problems for diverse objectives.

3.1. Radiocoloring

The notion of radiocoloring is a coloring conception convenient to modelize ordinary frequency assignment problems. A graph G is radiocolored if the colors $c(v_i)$ assigned to every node $v_i \in V$ verify the following two conditions

i) $\left| c(v_i) - c(v_j) \right| \geq 2$ for every $(v_i, v_j) \in E$

ii) if $d(v_i, v_j) = 2$ then $c(v_i) \neq c(v_j)$

Namely, if two stations interfere, i.e., $(v_i, v_j) \in E$ then the frequency difference must be greater or equal than 2. Also stations that are distance two apart should not be assigned the same color, this means that the intermediate nodes of the couple $\{v_i, v_j\}$ when $d(v_i, v_j) = 2$ must not be influenced by the same frequency, as shown in Figure 4. The squared nodes are intermediate nodes of the couple $\{v_i, v_j\}$ for which $d(v_i, v_j) = 2$.

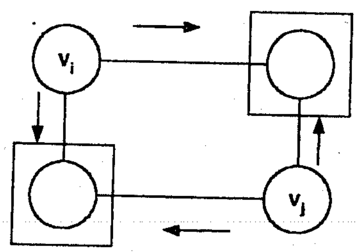

Figure 4. *Intermediate nodes*

A radiocoloring that uses q colors i a $q-$radiocoloring . Let $G = (V, E)$ be a graph with m nodes. We represent a particular radiocoloring, say R, with the use of a linear array of m elements, where the i element of R express the color assigned to node v_i, namely, $R(i) = c(v_i)$.

The following invariants can be defined in a graph G:

- The *radiochromatic score* $rs(G, R)$ of a radiocoloring R is the number of used colors.
- The number of colors used in a radiocoloring with the minimum score is the *radiochromatic number* $rn(G)$ of G.
- The *radiochromatic price* $rp(G, R)$ of a radiocoloring R is the value of the largest used color.
- The largest used color in a radiocoloring with the minimum price is the *radiochromatic value* $rv(G)$ of G.
- The *radiochromatic weight* $rw(G, R)$ of a radiocoloring R is the sum of the used colors.
- The sum of the used colors in a radiocoloring with the minimum weight is the *radiochromatic cost* $rc(G)$ of G.
- The *radiochromatic gap* $rg(G, R)$ of a radiocoloring R is the difference between the smallest and the largest used colors.
- The difference between the smallest and the largest used colors in a radiocoloring with the minimum gap is the *radiochromatic bandwidth* $rb(G)$ of G.

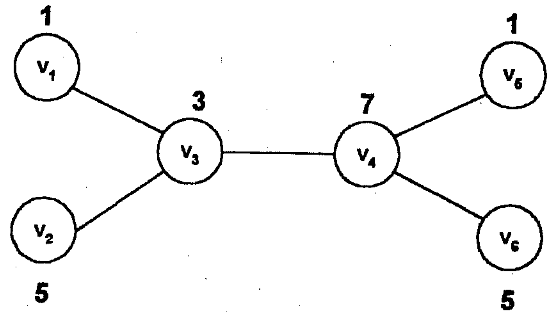

Figure 5. *4* – *radiocoloring*, $R=[1\ 5\ 3\ 7\ 1\ 5]$

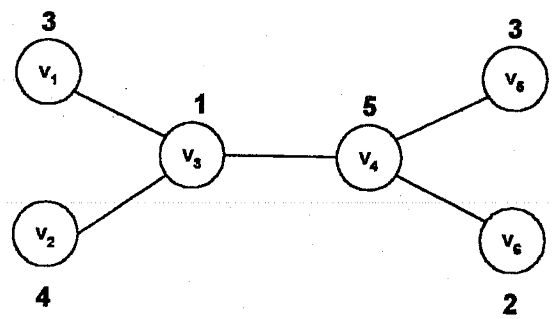

Figure 6. *5* – *radiocoloring*, $R=[3\ 4\ 1\ 5\ 3\ 2]$

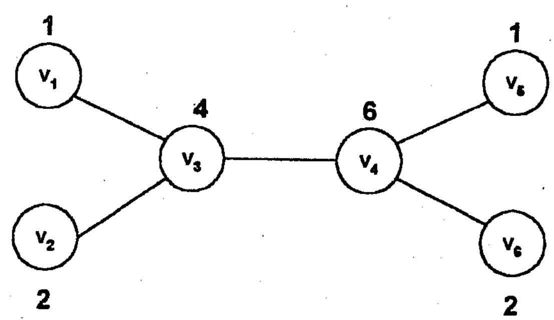

Figure 7. *4* – *radiocoloring*, $R=[1\ 2\ 4\ 6\ 1\ 2]$

The Figures 5,6 and 7 show two radiocolorings of a six nodes graph G. The values of the invariants for the radiocoloring of Figure 5 are

$\boldsymbol{rs(G)=rn(G)=4}$, $rp(G)=7$, $rw(G)=22$, $rg(G)=6$.

For the radiocoloring of Figure 6 we have

$rs(G)=5$, $\boldsymbol{rp(G)=rv(G)=5}$, $rw(G)=18$, $\boldsymbol{rg(G)=rf(G)=4}$

and for the radiocoloring of Figure 7

$\boldsymbol{rs(G)=rn(G)=4}$, $rp(G)=6$, $\boldsymbol{rw(G)=rc(G)=16}$, $rg(G)=5$.

The relations reflected with bold character indicate the minimum values of the corresponding invariant.

3.2. $T-$ coloring

Let T be a set that contains 0 and positive integers. A coloring of $G=(V,E)$ is called a $T-$*coloring* if the difference of the colors assigned to adjacent nodes of G is not in T, i.e., $\left| c(v_i)-c(v_j) \right| \notin T$.

Two transmission stations that utilize Ultra-High Frequencies (UHF) and that interfere should use frequencies the difference of which must not belong to set $T=\{0,7,14,15\}$.

A $T-$coloring that uses q colors is a $q-T-$*coloring*. A particular $T-$coloring is represented by an array with the same manner as for radiocoloring. Similarly to radiocoloring the subsequent invariant for the $T-$coloring of G can be deduced.

The following invariants can be defined in a graph G:

- The $T-$*chromatic score* $Ts(G,T,R)$ of a $T-$coloring is the number of used colors.
- The number of colors used in a $T-$coloring with the minimum score is the $T-$*chromatic number* $Tn(G,T)$ of G.
- The $T-$*chromatic price* $Tp(G,T,R)$ of a $T-$coloring is the value of the largest used color.
- The largest used color in a $T-$coloring with the minimum price is the $T-$*chromatic value* $Tv(G,T)$ of G.
- The $T-$*chromatic weight* $Tw(G,T,R)$ of a $T-$coloring is the sum of the used colors.
- The sum of the used colors in a $T-$coloring with the minimum weight is the $T-$*chromatic cost* $Tc(G,T)$ of G.
- The $T-$*chromatic gap* $Tg(G,T,R)$ of a $T-$coloring is the difference between the smallest and the largest used colors.
- The difference between the smallest and the largest used colors in a $T-$coloring with the minimum gap is the $T-$*chromatic bandwidth* $Tb(G,T)$ of G.

The $T-$coloring for the sketched graph in Figure 8 for $T=\{0,1,3\}$, corresponds to a $T-$coloring, that corresponds to the following values of the invariant described above $Tn(G)=3$, $Tv(G)=5$, $Tc(G)=18$ and $Tb(G)=4$.

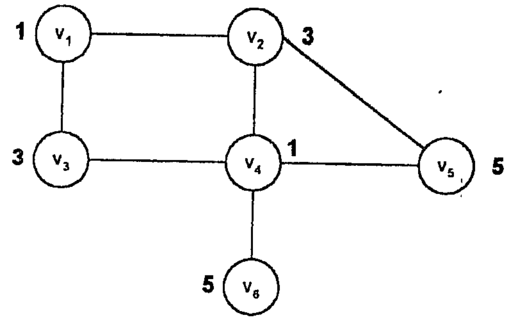

Figure 8. $T-coloring$, $R = [1\ 3\ 3\ 1\ 5\ 5]$

Of course for larger graphs it is unusual for a specific $T-$coloring to meet simultaneously all these $T-$coloring invariants. The next section is devoted to a $T-$ coloring algorithm.

4. $T-$COLORING ALGORITHM

Here we will develop a heuristic algorithm that generates a $T-$coloring that has a satisfactory approximation of the $T-$chromatic number and the $T-$chromatic cost for an arbitrary graph G. Before proceeding to the steps of the algorithm we give the interpretation of the used notations and a description of the algorithm reasonings.

K	:	Set that contains the colors used so far
C_j	:	Set that contains the nodes colored with color $j \in K$
U	:	Set that contains the so far uncolored nodes $U \subseteq V$
$VU = V - U$	:	set that contains the so far colored nodes
$H(U,v)$	:	A maximal independent set in subgraph $G(U)$ produced by $v \in U$, i.e. $v \in H(U,v)$
$\Gamma(H(U,y)) \cap VU$	:	colored nodes adjacent to $H(U,y)$

The algorithm proceeds in a greedy way. Specifically, the process finds an uncolored node, say y, that leads to a large maximal independent set (*mis*) $H(U,y)$ in subgraph $G(U)$ of G. Subsequently the nodes of $H(U,y)$ are colored with the smallest feasible color, say j, feasible in the sense that the new color assigned to the nodes of $H(U,y)$ must not violate the $T-$coloring condition. In other words the colors of the colored adjacent nodes of $H(U,y)$ which is the set $\Gamma(H(U,y)) \cap VU$ must have a separation that, is not contained in T. The algorithm terminates when $U = \varnothing$.

The above analysis is systematically applied in the following steps.

T – coloring Algorithm

Step 1 : Set $U \leftarrow V$, $K \leftarrow \varnothing$

Step 2 : If $U = \varnothing$ go to Step 6

Step 3 : Find node y such that $H(U,y) = \max_{v \in U}\{ H(U,v) \}$

Step 4 : Let $j = \{$ the smallest positive integer such that $|\, j - c(w) \,| \notin T$, $\forall w \in (\Gamma(H(U,y)) \cap VU) \}$

Step 5 : Set $C_j \leftarrow H(U,y), K \leftarrow K + \{ j \}, U \leftarrow U - H(U,y)$ and proceed to Step 2

Step 6 : Write

i. T – chromatic price $|K|$

ii. T – chromatic weight $\sum_{j \in K} |C_j|$

iii. The T – coloring class C_j, $j \in K$

<u>Example</u>

Consider the graph $G = (V,E)$ of Figure 9 and $T = \{ 0, 1, 3 \}$

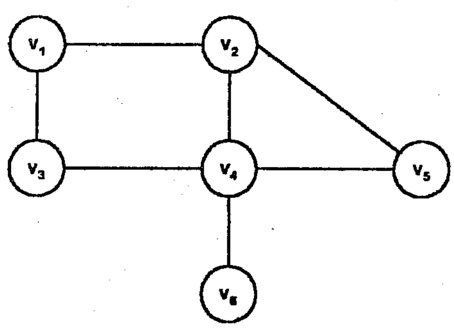

Figure 9. *Graph G*

We form the list array A of nonadjacent nodes for every node $v \in U$. At the starting step $U = V$. The nonadjacent nodes are placed from left to right in increasing order of their degree in $G(U)$ so to augment the probability to achieve a large maximal independent set, since the construction of a *mis* is performed from left to right. In the following tables the nodes are indicated by their indices.

For $U = V$

A=

i	$\overline{\Gamma}_U(v_i)$			
1	6	5	4	
2	6	3		
3	6	5	2	
4	1			
5	6	1	3	
6	5	3	1	2

i	$H(U,v_i)$		
1	1	6	5
2	2	6	3
3	3	6	5
4	4	1	
5	5	6	1
6	6	5	3

A largest *mis* among alternatives is $\{1, 6, 5\}$. We color these nodes with the color 1, thus $C_1 = \{1, 6, 5\}$.

We get the subgraph of the remaining uncolored node $U' = \{2, 3, 4\}$. The subgraph $G(U')$ is showed with bold lines in Figure 10. The assigned colors are indicated nearby each colored node.

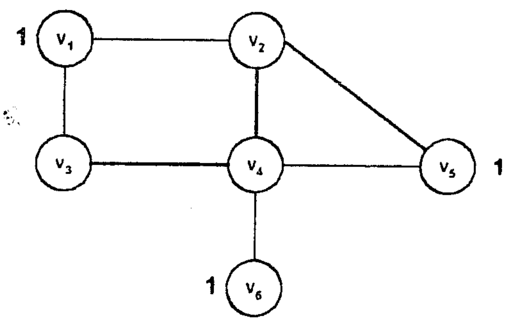

Figure 10. *Subgraph* $G(U')$*, in bold.* $R = [1 - - - 1\ 1]$

For $U' = \{2, 3, 4\}$

A=

i	$\overline{\Gamma}_i(v_i)$
2	3
3	2
4	2 3

i	$H(U', v_i)$
2	2 3
3	3 2
4	4

We color node 2 and 3 with the smallest feasible color, which is color number 3. We get the one node subgraph, shown in bold.

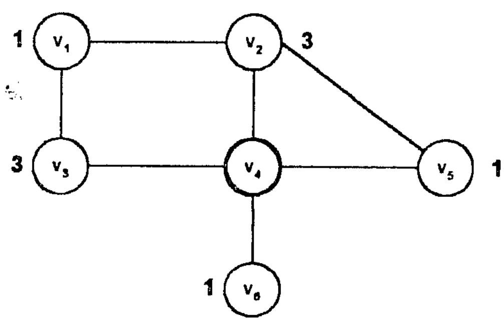

Figure 11. *Subgraph* $G(U')$, $U' = \{4\}$, $R = [1\ 3\ 3 - 1\ 1]$

For $U' = \{4\}$

A=

i	$\overline{\Gamma}_i(v_i)$
4	$\emptyset$

i	$H(U', v_i)$
4	4

We color node 4 with color 5.

The algorithm terminates since $U' = \{\emptyset\}$. The derived T-coloring of G is :

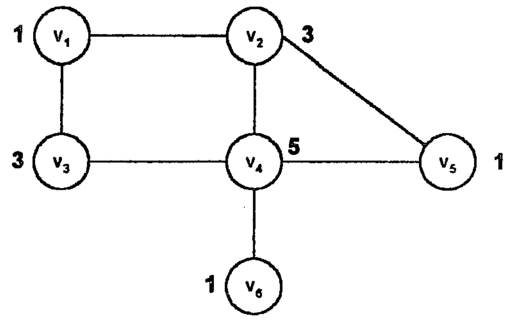

Figure 12. *Derived T – coloring,* $R = [1\ 3\ 3\ 5\ 1\ 1]$

5. CONCLUSIONS

The radiocoloring and T – coloring are reduced to the simple coloring with a casual alteration in their definitions. The radiocoloring is equivalent to coloring if the condition of distance 2 is dropped and the first condition requires that the color separation of adjacent nodes is greater or equal than 1 instead of 2, that is

$$\left| c(v_i) - c(v_j) \right| \geq 1 \quad , \forall (v_i, v_j) \in E$$

The T – coloring is identical to the simple coloring if the positive integers are subtracted from T, that is when $T = \{ 0 \}$.

The concepts of radiocoloring and T – coloring and the corresponding invariants with minor modification can be suitable for the solving problems in diverse fields, as for example resource or job allocation, task scheduling, investment planning, etc. where the nodes may represent locations, equipments, profit, etc. and the edges constraints concerning simultaneous utilization of investments, operation delay, etc.

All the invariants mentioned here belongs to the category of NP – complete problems [6] and the corresponding minimization problems are NP – Hard.

For relatively large graphs, it is laborious even to find a feasible solution for the T – coloring problem without the use of a relative computer algorithm. The procedure presented in section 4 gives an approximation of the T – chromatic number and T – cost in addition of a feasible T – coloring for an arbitrary graph.

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