The Journal of
MANAGEMENT SCIENCES &
REGIONAL DEVELOPMENT
Issue 3, July 2001

AN INVENTORY MODEL WITH DETERIORATING ITEMS, TIME-VARYING DEMAND AND PARTIALLY TIME-VARYING BACKLOGGING-TAKING ACCOUNT OF TIME VALUE

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Abstract. We study a continuous review inventory model over a finite planning horizon with deterministic time-varying demand and constant deterioration rate. The model allows for shortages, which are partial backlogged at a rate that varies exponentially with time. The effect of time value of money is, also, taken into consideration. Applying the discount cash flow (DCF) approach for problem analysis, we propose an algorithm to find the optimal replenishment policy for this model.

Keywords. Inventory; deterioration; partial backlogging; inflation; time discounting.

1. INTRODUCTION

The deterioration of many items during storage period is a real fact. Ghare and Schrader [1] have studied an inventory model taking into account the effect of deterioration of items in storage. In their model they introduced a constant deterioration rate, while demand rate was also taken to be constant. Covert and Phillip [2], Tadikamalla [3], and Shah [4] extended Ghare and Schrader's model by relaxing the assumption of constant deterioration rate and introducing other modes of deterioration.

Later, Dave and Patel [5] developed an inventory model with deterministic but linearly changing demand rate, constant deterioration rate and finite planning horizon. Sachan [6] extended Dave and Patel's model to allow for shortages. Datta and Pall [7] presented an EOQ model considering variable deterioration and power demand pattern. Research on models with deteriorating items, time varying demand and shortages continues with Goswami and Chaudhuri [8], Benkherouf [9], Hariga [10], Chakrabarti and Chaudhuri [11], Hariga and Alyan [12], Teng et al [13]. The common characteristic of all the above articles is that they allow for shortages while unsatisfied demand is completely backlogged.

Wee and Mercan [14] relaxed this assumption. They considered an inventory model over a finite-planning horizon with constant demand and deterioration rates. Additionally they assumed that only a fraction β, $0<\beta<1$, of demand during the stockout period is backlogged. Other researchers who used the idea of partial backlogging are Wee [15, 16], Chang and Dye [17] and others.

In all of the above-mentioned models, the time value of money was disregarded. However, with today's large scale of inflation in world economics, the impact of both inflation and time value of money on the selection of the inventory policy should be considered. The first attempt in this direction was the article by Buzacott [18] going back to 1975. Up to now a considerable volume of research concerning deteriorating inventories and time value of money has appeared in the literature. Sample of those are the articles [19]-[22].

In this article we study an inventory model over a finite planning horizon, with constant deterioration rate, time varying demand rate and time dependent partial backlogging. More explicitly we suppose that the rate of backlogged demand increases exponentially as the waiting time to the next replenishment decreases. We believe that this is a quite reasonable assumption since as the waiting time decreases, more and more customers are willing to wait to get their orders as soon as the backlogged demand reaches the system at the next replenishment. Moreover, we adopt a discounted cash flow approach to obtain the present value of involved costs. This is also reasonable approach since, owing to Asia, Russia and Brazil financial crisis, most of the countries have suffered from large scale inflation and sharp decline in the purchasing power of money last several years (see also Chung and Lin [21]).

2. ASSUMPTIONS AND NOTATION

In this section we state the assumptions under which our continuous review inventory model is developed and we present the notation used throughout this paper.

Assumptions

1. The planning horizon of the system is finite and is taken as H time units. The initial and final inventory levels during the planning horizon are both set to zero.
2. Replenishment is instantaneous (replenishment rate is infinite).
3. The lead-time is zero.
4. The on hand inventory deteriorates at a constant rate θ ($0<\theta<1$) per time unit. The deteriorated items are withdrawn immediately from the warehouse and there is no provision for repair or replacement.
5. The rate of demand at time, $t \in [0, H]$, is a continuous, logconcave function $f(t)$ with $f'(t)\neq 0$ for all t.
6. The system allows for shortages in all cycles and each cycle starts with shortages.
7. Unsatisfied demand is backlogged at a rate $\exp(-\alpha x)$, where x is the time up to the next replenishment and α a parameter.
8. A DCF approach is adopted to consider the time value of money. The discount is applied continuously.

Notation

n the number of replenishment cycles during the planning horizon.
s_i time at which shortage starts during the *i*th cycle $i=1,\ldots,n$.
t_i time at which the *i*th replenishment is made $i=1,\ldots,n$.
q_i variable replenishment lot size ordered at instant t_i $i=1,\ldots,n$.
$A + Cq_i$ the replenishment cost for the *i*th cycle, where A is a fixed set up replenishment cost and C is the additional replenishment cost paid per unit of ordered item.
C_1 holding cost per unit of stock carried per unit time.
C_2 shortage cost per unit of shortage per unit time.
C_3 opportunity cost due to lost sales per unit of lost sale
r discount rate
CI_i the amount of inventory carried during the *i*th cycle.
DI_i the amount of deteriorated items during the *i*th cycle.
SI_i the amount of units in shortages during the *i*th cycle.
LI_i the amount of lost sales units during the *i*th cycle.
$I(t)$ the inventory level at time t.

3. THE MATHEMATICAL FORMULATION OF THE MODEL

A realization of the inventory level in the system is given in figure 1. The depletion of the inventory level during the interval $[t_i, s_i]$, of the ith replenishment cycle, is due to the joint effect of demand and deterioration. Hence the differential equation which describes the variation of inventory level, I(t), with respect to time t is:

$$\frac{dI(t)}{dt} = -\theta I(t) - f(t), \quad t_i \le t < s_i \tag{1}$$

with boundary condition $I(s_i)=0$, $i=1,\ldots,n$.

The solution of (1) is

$$I(t) = e^{-\theta t} \int_t^{s_i} e^{\theta u} f(u)du, \quad t_i \le t < s_i. \tag{2}$$

From (2) we get that, the amount of inventory carried during the ith cycle is given by

$$CI_i = \frac{1}{\theta} \int_{t_i}^{s_i} (e^{\theta(t-t_i)} - 1) f(t)dt. \tag{3}$$

The variation of the inventory level, I(t), at any time, t, in the interval $[s_{i-1}, t_i]$ is described by the differential equation

$$\frac{dI(t)}{dt} = -e^{-\alpha(t_i - t)} f(t), \quad s_{i-1} \le t < t_i \tag{4}$$

with boundary condition $I(s_{i-1})=0$ $i=1,\ldots n$

The solution of (4) is

$$I(t) = -\int_{s_{i-1}}^{t} e^{-\alpha(t_i - u)} f(u)du, \quad s_{i-1} \le t < t_i \tag{5}$$

From (5) we obtain the amount of shortage during the ith cycle as

$$SI_i = \int_{s_{i-1}}^{t_i} \int_{s_{i-1}}^{t} e^{-\alpha(t_i - u)} f(u)dudt = \int_{s_{i-1}}^{t_i} e^{-\alpha(t_i - u)} (t_i - u) f(u)du. \tag{6}$$

The amount of lost sales during the ith cycle is

$$LI_i = \int_{s_{i-1}}^{t_i} (1 - e^{-\alpha(t_i - u)}) f(u)du. \tag{7}$$

The lot size, q_i, for the ith cycle, is made up by the sum of maximum and minimum stock levels recorded over the cycle. So by, using equations (5) and (2), we obtain it as

$$q_i = \int_{s_{i-1}}^{t_i} e^{-\alpha(t_i - u)} f(u)du + \int_{t_i}^{s_i} e^{\theta(u - t_i)} f(u)du. \tag{8}$$

In order to formulate the present value of the total inventory cost function we need to find the present value of each of the involved cost during the *ith* cycle.
The present value of the purchasing cost during the *ith* cycle is given by

$$Ce^{-rt_i}\left(\int_{s_{i-1}}^{t_i} e^{-\alpha(t_i-u)}f(u)du + \int_{t_i}^{s_i} e^{\theta(u-t_i)}f(u)du\right) \tag{9}$$

The present value of the holding cost during the *ith* cycle is given by

$$C_1\int_{t_i}^{s_i} e^{-(\theta+r)t}\int_{t}^{s_i} e^{\theta u}f(u)dudt \tag{10}$$

Using integration by parts we obtain

$$\frac{C_1e^{-rt_i}}{r+\theta}\int_{t_i}^{s_i}(e^{\theta(u-t_i)} - e^{-r(u-t_i)})f(u)du \tag{11}$$

The present value of the shortages cost during the *ith* cycle is given by

$$C_2\int_{s_{i-1}}^{t_i} e^{-rt}\int_{s_{i-1}}^{t} e^{-\alpha(t_i-u)}f(u)dudt \tag{12}$$

which simplifies to

$$\frac{C_2}{r}\int_{s_{i-1}}^{t_i} e^{-\alpha(t_i-u)}(e^{-ru} - e^{-rt_i})f(u)du \tag{13}$$

The present value of the lost sales cost during the *ith* cycle is given by

$$\int_{s_{i-1}}^{t_i} e^{-ru}(1-e^{-\alpha(t_i-u)})f(u)du \tag{14}$$

Now we have all necessary quantities to formulate the present value of the total inventory cost function for any policy, which has n cycles. This is the sum of the following costs: replenishment, holding, shortage and opportunity, and is given by

$$TC(n, s_i, t_i) = \sum_{i=1}^{n}(A+Cq_i) + C_1\sum_{i=1}^{n}CI_i + C_2\sum_{i=1}^{n}SI_i + C_3\sum_{i=1}^{n}LI_i$$

$$= \sum_{i=1}^{n}Ae^{-rt_i} + \sum_{i=1}^{n}\int_{t_i}^{s_i}\left(\frac{C_1e^{-rt_i}}{r+\theta}(e^{\theta(u-t_i)} - e^{-r(u-t_i)}) + Ce^{-(r+\theta)t_i+\theta u}\right)f(u)du$$

$$+\sum_{i=1}^{n}\int_{s_{i-1}}^{t_i}\left(Ce^{-rt_i}e^{-\alpha(t_i-u)} + \frac{C_2}{r}e^{-\alpha(t_i-u)}(e^{-ru} - e^{-rt_i}) + C_3e^{-ru}(1-e^{-\alpha(t_i-u)})\right)f(u)du \tag{15}$$

The minimization of this function w.r.t. n, t_i, s_i, will give the optimal policy of the problem.

4. THE OPTIMAL REPLENISHMENT POLICY

The continuity of f(t) implies that $TC(n, s_i, t_i)$ is a continuous function of s_i, t_i and its first and second order partial derivatives exist. For a fixed value of n, taking first order derivatives of $TC(n, s_i, t_i)$ with respect to t_i, s_i and equating them to zero we obtain the necessary conditions for a minimum of $TC(n, s_i, t_i)$, which are:

$$Ar + (C_1 + C(r+\theta)\int_{t_i}^{s_i} e^{\theta(u-t_i)} f(u)du =$$

$$\int_{s_{i-1}}^{t_i} (e^{-\alpha(t_i-u)}(-C(r+\alpha) - \frac{\alpha C_2}{r}(e^{r(t_i-u)} - 1) + C_2 + C_3\alpha e^{r(t_i-u)})f(u)du, \ i=1,\dots,n \quad (16)$$

$$Ce^{-(r+\theta)t_i+\theta s_i} + \frac{C_1}{r+\theta}[e^{\theta(s_i-t_i)} - e^{-r(s_i-t_i)}] =$$

$$e^{-\alpha(t_{i+1}-s_i)}(Ce^{-rt_{i+1}} + \frac{C_2}{r}(e^{-rs_i} - e^{-rt_{i+1}}) - C_3e^{-rs_i}) + C_3e^{-rs_i}, \ i=1,\dots,n-1 \quad (17)$$

We must note that the above systems of equations has always a solution if the following conditions are satisfied:

a) $C_2+C_3\alpha-C(r+\alpha) - \frac{C_2\alpha}{r}(e^{rH} - 1) > 0$

b) $C > \frac{C_2}{r} > C_3$

That is the above conditions are sufficient but not necessary for the existence of a solution.

The first condition guarantees that the right hand side of the eq. (16) is a positive number and the second condition guarantees that the right hand side of the eq. (17) is a positive number.

If t_i^*, s_i^*, i=1, 2,...,n, is a solution of the above system of equations, this gives a minimum for TC(n, s_i, t_i). The proof is given in the appendix. Moreover we shall prove that the system of equations (16), (17) has a unique solution in t_i, s_i on the interval [0, H] and so the above minimum is a global minimum.

We shall now present the algorithm used to solve the system of equations (16) and (17) and we shall prove that the obtained solution is unique. It is easy to see that, once t_1 is known, $s_1(t_1)$ can be obtained from (16). Then $t_2(t_1)$ can be obtained from (17) and following this alternate procedure we can find $s_2(t_1)$,..., $s_n(t_1)$. The choice of t_1 plays a key role, as all subsequent parameters become functions of this. It is obvious that the set of t_i, s_i values so produced will be a solution of the above system only if the starting point t_1 is such that $s_n(t_1)$=H. So the question, which arises is if such a choice for t_1 is possible. The lemma, which follows, gives a positive answer.

Lemma 1. The function $s_n(t_1)$ is monotonically increasing with respect to the variable t_1 if the demand rate is an increasing function w.r.t. t.

Proof. Since f(t) is logconcave, $\frac{f(t)}{f'(t)}$ is strictly increasing in t, for $t_i \le t \le s_i$, and so we have $f'(t) \le \frac{f'(t_i)}{f(t_i)} f(t)$. Multiplying this inequality by $e^{\theta(t-t_i)}$ and adding the term $\theta e^{\theta(t-t_i)} f(t) > 0$ to both sides of the resulting inequality, we obtain

$$e^{\theta(t-t_i)}f'(t) + \theta e^{\theta(t-t_i)}f(t) \le e^{\theta(t-t_i)}\frac{f'(t_i)}{f(t_i)}f(t) + \theta e^{\theta(t-t_i)}f(t), \ t_i \le t \le s_i. \quad (18)$$

Now we multiply both sides of (18) by $C_1+C(r+\theta)$ and then we integrate with respect to t in the interval $[t_i, s_i]$. After some elementary operations we obtain the inequality

$$(C_1+C(r+\theta))[e^{\theta(s_i-t_i)}f(s_i)-f(t_i)]\leq(C_1+C(r+\theta))[\frac{f'(t_i)}{f(t_i)}+\theta]\int_{t_i}^{s_i}e^{\theta(t-t_i)}f(t)dt. \quad (19)$$

Due to (16) the above inequality becomes

$$(C_1+C(r+\theta))[e^{\theta(s_i-t_i)}f(s_i)-f(t_i)-\theta\int_{t_i}^{s_i}e^{\theta(t-t_i)}f(t)dt]$$
$$\leq\frac{f'(t_i)}{f(t_i)}(-Ar+\int_{s_{i-1}}^{t_i}(e^{-\alpha(t_i-u)}(-C(r+\alpha)-\frac{\alpha C_2}{r}(e^{r(t_i-u)}-1)+C_2+C_3\alpha e^{r(t_i-u)})f(u)du, \quad (20)$$

We know that

$$C_2+C_3\alpha-C(r+\alpha)-\frac{C_2\alpha}{r}(e^{rH}-1)>0 \text{ and } \frac{f'(t_i)}{f(t_i)}<\frac{f'(t)}{f(t)} \text{ for } s_{i-1}\leq t\leq t_i.$$

Taking these inequalities into account in the right hand side of (20) and then integrating by parts we obtain

$$\begin{aligned}&(C_1+C(r+\theta))[e^{\theta(s_i-t_i)}f(s_i)-f(t_i)-\theta\int_{t_i}^{s_i}e^{\theta(t-t_i)}f(t)dt]\leq\\&+e^{-\alpha(t_i-s_{i-1})}(-C(r+\alpha)-\frac{\alpha C_2}{r}(e^{r(t_i-s_{i-1})}-1)+C_2+\alpha C_3e^{r(t_i-s_{i-1})})f(s_{i-1})\\&+(-(C(r+\alpha)+C_2+C_3\alpha)f(t_i)-Ar\frac{f'(t_i)}{f(t_i)}\\&-\int_{s_{i-1}}^{t_i}\alpha e^{-\alpha(t_i-u)}(-C(r+\alpha)-\frac{\alpha C_2}{r}(e^{r(t_i-u)}-1)+C_2+\alpha C_3e^{r(t_i-u)})f(u)du\end{aligned} \quad (21)$$

Let us set $M_i=s_i-t_i$ and $K_i=t_i-s_{i-1}$. Obviously M_i, K_i are functions of t_1. Substituting M_i, K_i into (16) and differentiating (16) with respect to t_1 for $i=1,\ldots,n$, we obtain the system of equations.

$$\begin{aligned}&(C_1+C(r+\theta))\{e^{\theta M_i}f(s_i)\frac{dM_i}{dt_1}+\frac{dt_i}{dt_1}[e^{\theta M_i}f(s_i)-f(t_i)-\theta\int_{t_i}^{s_i}e^{\theta(t-t_i)}f(t)dt]\}\\&=(-C(r+\alpha)+C_2+\alpha C_3)\frac{dt_i}{dt_1}f(t_i)\\&-(\frac{dt_i}{dt_1}-\frac{dK_i}{dt_1})(-C(r+\alpha)+C_2-\frac{\alpha C_2}{r}(e^{rK_i}-1)+\alpha C_3e^{rK_i})e^{-\alpha K_i}f(s_{i-1})\\&-\alpha\frac{dt_i}{dt_1}\int_{s_{i-1}}^{t_i}e^{-\alpha(t_i-u)}(-C(r+\alpha)-\frac{\alpha C_2}{r}(e^{r(t_i-u)}-1)+C_2+\alpha C_3e^{r(t_i-u)})f(u)du\end{aligned} \quad (22)$$

For $i=1$ we have $K_1=t_1-s_0=t_1$ and since $\frac{dt_1}{dt_1}=1$ we also have $\frac{dK_1}{dt_1}=1$. By using (22) for $i=1$ and replacing these derivatives we obtain

$$
\begin{aligned}
&(C_1+C(r+\theta))e^{\theta M_1}f(s_1)\frac{dM_1}{dt_1}= \\
&\frac{dK_1}{dt_1}e^{-\alpha K_1}(-C(r+\alpha)-\frac{\alpha C_2}{r}(e^{rK_1}-1)+C_2+\alpha C_3 e^{rK_1})f(s_0) \\
&+(-C(r+\alpha)+\alpha C_3+C_2)f(t_1) \\
&-e^{-\alpha K_1}(-C(r+\alpha)+C_2-\frac{\alpha C_2}{r}(e^{rK_1}-1)+\alpha C_3 e^{rK_1})f(s_0) \\
&-\alpha\int_{s_0}^{t_1}e^{-\alpha(t_1-u)}(-C(r+\alpha)-\frac{\alpha C_2}{r}(e^{r(t_1-u)}-1)+C_2+\alpha C_3 e^{r(t_1-u)})f(u)du \\
&-(C(r+\theta)+C_1)[f(s_1)e^{\theta M_1}-f(t_1)-\theta\int_{t_1}^{s_1}e^{\theta(t-t_1)}f(t)dt]
\end{aligned}
\tag{23}
$$

Using inequality (20) and the fact that $C_2+C_3\alpha-C(r+\alpha)-\frac{C_2\alpha}{r}(e^{rH}-1)>0$ we conclude that the right-hand side of (23) is a positive number, which implies that $\frac{dM_1}{dt_1}>0$.

Next, differentiating (17) with respect to t_1 we have

$$(C(\theta+r)+C_1)e^{\theta M_1}\frac{dM_1}{dt_1}=(-C(r+\alpha)-\frac{C_2}{r}(e^{rK_2}-1)+\alpha C_3+C_2)\frac{dK_2}{dt_1}e^{-(\alpha+r)K_2}$$

The left-hand side in the above equation is positive and so, $\frac{dK_2}{dt_1}>0$.

Using the same technique we can prove that $\frac{dK_i}{dt_1}>0$ and $\frac{dM_i}{dt_1}>0$ for $i=1,\ldots,n$. It is obvious that $s_n(t_1)=\sum_{i=1}^{n}(M_i+N_i)$. The above results show that $\frac{ds_n(t_1)}{dt_1}>0$, which implies that the function $s_n(t_1)$ is strictly increasing with respect to t_1.

Using induction on i we can prove the following

Corollary 1. The functions $t_i(t_1)$, $s_i(t_1)$, $i=1, 2,\ldots, n$ are monotonically increasing w.r.t. t_1 if $f(t)$ is a increasing function of t.
It is obvious that the first choice of t_1 is arbitrary (usually selected as $t_1=H/n$), however because of lemma 1 the variation (increasing/decreasing) of values of t_1 are prescribed such that $s_n(t_1)=H$.

Now we present the theorem, which ensures the existence of a unique optimal number of replenishments, n, and consequently the existence of a unique optimal policy for the problem under consideration.

Theorem 1. The cost, $TC(n, s_i^*, t_i^*)$, of the optimal policy with n replenishments, is a convex function with respect to n.

Proof. The proof of this theorem will be established by using dynamic programming. The technique is similar to that used by Teng et al. [13] and Friedman [23]. A policy with n replenishment cycles and s_i, t_i shortages and ordering points, starts its first cycle at $s_0=0$ and ends its last cycle at time $s_n=H$. The realization of the stock level in the system is monitored and as the ith cycle is completed we observe the value of the decision variable s_i. This can be any value in the interval (0, H). So, for this process the stage space (points at which we keep records of the process) is {1, 2, ..., n}, while the state space (points at which shortages start) at each stage is any value in the interval (0, H)

To prove the convexity it is enough to show that

$$TC(n+1, 0, H) - TC(n, 0, H) > TC(n, 0, H) - TC(n-1, 0, H) \tag{24}$$

Applying Bellman's principle of optimality [24] we end up with the following forward dynamic programming equation

$$TC(n, 0, H) = \min_{s \in (0, H)} \{TC(n-1, 0, s) + G(1, s, H)\} \tag{25}$$

where G(1, s, H) is the cost of the last cycle starting at any time s and finishing at H.

Recursive application of (25) yields the optimal ith shortage point, $s^*_{n-j}(n, 0, H) = s^*_{n-j}(n-j+1, 0, s^*_{n-j+1}(n, 0, H))$, $j=1,\dots,n$, for any policy with n orders placed in the interval [0, H].

For s=H (25) gives TC(n, 0, H)<TC(n-1, 0, H). This proves that TC(n, 0, H) is strictly decreasing function with respect to n. Let us now choose H_1, H_2 >H so that

$$s^*_n(n+1, 0, H_1) = s^*_{n+1}(n+2, 0, H_2) = H. \tag{26}$$

Since $s^*_n(n+1, 0, H_1) = H$, by employing the principle of optimality, we have

$$\begin{aligned} TC(n+1, 0, H_1) &= \min_{s \in (0, H_1)} \{TC(n, 0, s) + G(1, s, H_1)\} \\ &= TC(n, 0, H) + G(1, H, H_1). \end{aligned} \tag{27}$$

But this implies that

$$\frac{\partial [TC(n, 0, t) + G(1, t, H_1)]}{\partial t} = \frac{\partial TC(n, 0, t)}{\partial t} + \frac{\partial G(1, t, H_1)}{\partial t}\Big|_{t=H} = 0. \tag{28}$$

So

$$\frac{\partial TC(n, 0, t)}{\partial t}\Big|_{t=H} = -\frac{\partial G(1, t, H_1)}{\partial t}\Big|_{t=H}$$

$$= \left(Ce^{\theta H - (r+\theta) t^*_n(n, 0, H)} + \frac{C_1}{r+\theta}\left(e^{\theta(H - t^*_n(n, 0, H))} - e^{-r(H - t^*_n(n, 0, H))}\right)\right) f(H). \tag{29}$$

where $t^*_n(n, 0, H)$ is the last optimal replenishment time when n orders are placed during the interval [0, H]. Similarly, from $s^*_{n+1}(n+2, 0, H_2) = H$, we have

$$\frac{\partial TC(n+1, 0, t)}{\partial t}\Big|_{t=H} = -\frac{\partial G(1, t, H_1)}{\partial t}\Big|_{t=H}$$

$$= \left(Ce^{\theta H - (r+\theta) t^*_{n+1}(n+1, 0, H)} + \frac{C_1}{r+\theta}\left(e^{\theta(H - t^*_{n+1}(n+1, 0, H))} - e^{-r(H - t^*_{n+1}(n+1, 0, H))}\right)\right) f(H). \tag{30}$$

Subtracting eq. (30) from eq. (29) we obtain

$$\frac{\partial[TC(n,\ 0,\ H)-TC(n+1,\ 0,\ H)}{\partial H}>0, \tag{31}$$

which implies that T(n, 0, H)-T(n+1, 0, H) is a strictly increasing function with respect to H. From (25) and (27) we obtain

$$TC(n, 0, H_1) - TC(n+1, 0, H_1) = \min_{s\in(0,\ H_1)} \{TC(n-1, 0, s)+ G(1, s, H_1)\} - \{TC(n, 0, H)+G(1, H, H_1)\}. \tag{32}$$

For s=H, eq. (32) gives

$$TC(n, 0, H_1) - TC(n+1, 0, H_1)< TC(n-1, 0, H) - TC(n, 0, H) \tag{33}$$

Since $H<H_1$ and T(n, 0, H)-T(n+1, 0, H) is a strictly increasing function with respect to H, we obtain the relation (24). This implies that $TC(n,\ s_i^*,\ t_i^*)$ is also convex with respect to n.

5. NUMERICAL EXAMPLES

To illustrate the algorithm described above we consider the following examples.

Example 1

$f(t)=600+2t$, $A=250$, $C=5$, $C_1=1.75$, $C_2=3$, $C_3=4$, $r=0.2$, $\theta=0.2$, $\alpha=0.02$, $H=10$.

From [16] and [17] and $s_n(t_1)=H$ the present value of total cost, TC, can be found for different values of n. In table 1 we give the optimal replenishment policy. Its corresponding cost is 16371.6.

n=7	t_i	0.4815	1.2461	2.1132	3.1167	4.5252	6.3777	9.0764
	s_i	0.7267	1.5098	2.4177	3.5406	5.0086	7.0318	10

Table 1: The optimal replenishment policy.

For $\alpha=0$ this example results to the one used by Chung and Tsai [22].

Example 2

We consider the same as above example changing the demand rate to $f(t)=20e^{0.5t}$. In table 2 we give the optimal replenishment policy. Its corresponding cost is 8078.8.

n=5	t_i	2.76	4.0212	6.6057	8.1242	9.6274
	s_i	3.3642	5.3895	7.0117	8.5082	10

Table 2: The optimal replenishment policy.

6. CONCLUDING REMARKS

1. In the model studied in this paper if we set the parameter α, of the backlogging rate function equal to zero, i.e. $\alpha=0$, we revert to the model with complete backlogging of the unsatisfied demand.
2. If $\alpha=0$ and $f(t)$ linear we obtain the model studied by Bhunia and Maiti [25]. However we must mention that the optimal policy found by these authors belongs to the class of policies with replenishment cycles of equal lengths, while in our model we have relaxed this condition by allowing policies with replenishment cycles of any length.
3. If we set $C=0$, $\alpha=0$ and $\theta=0$ we obtain the model developed by Hariga [10].
4. If we set $C=0$, $\alpha=0$, $\theta=0$ and $f(t)$ linear, our model yields the one studied by Goyal et al [26].
5. If we set $C=0$, $C_3=0$, $r=0$, we obtain the model developed Papachristos and Skouri [27].
6. If we set $r=0$, we obtain the model developed Skouri and Papachristos [28].
7. If $f(t)$ is constant we obtain the model developed by Chung and Lin [21].
8. If $f(t)$ is linear we obtain the model developed by Chung and Tsai [22].

However the most of the above mentioned models considered the deterioration cost but we believe that this cost can be embedded in the holding cost.

APPENDIX

Checking the conditions for a minimum of $TC(n, s_i, t_i)$.

For convenience let us set $TC(n, s_i, t_i)=TC$. To ensure that the solution of equations (16) and (17) gives a minimum it is sufficient to prove that the associated Hessian matrix has all its principal minors positive. Thereinafter we prove that this is true if f(t) is an increasing function of t. The elements of the Hessian matrix are.

$$H_{2k+1,2j+1}=\frac{\partial^2 TC}{\partial t_{k+1}\partial t_{j+1}} \quad j, k=0, 1,\dots, n-1,$$

$$H_{2k,2j}=\frac{\partial^2 TC}{\partial s_k \partial s_j} \quad k, j=1,\dots,n-1,$$

$$H_{2k+1,2j}=\frac{\partial^2 TC}{\partial t_{k+1}\partial s_j} \quad k=0, 2, \dots, n-1, j=1,\dots,n,$$

$$H_{2k,2j+1}=\frac{\partial^2 TC}{\partial s_k \partial t_{j+1}} \quad k=1,\dots,n, j=1,2, \dots, n-1.$$

The non - zero entries of the Hessian matrix are

$$H_{2j+1,2j+1}=\frac{\partial^2 TC}{\partial t_{j+1}^2}=$$

$$(C(r+\theta)+C_1)f(t_{j+1})+\theta(C(\theta+r)+C_1)\int_{t_{j+1}}^{s_{j+1}} e^{\theta(t-t_{j+1})}f(t)dt+(-C(r+\alpha)+\alpha C_3+C_2)f(t_{j+1})$$

$$-\int_{s_j}^{t_{j+1}} e^{-\alpha(t_{j+1}-u)}(-C(r+\alpha)-\frac{\alpha C_2}{r}(e^{r(t_{j+1}-u)}-1)+C_2+\alpha C_3 e^{r(t_{j+1}-u)}+(C_2-rC_3)e^{r(t_{j+1}-u)})f(u)du$$

$$H_{2j,2j}=\frac{\partial^2 TC}{\partial s_j^2}=(C_1+C(r+\theta))\,e^{(r+\theta)(s_j-t_j)}e^{-rt_j}\,f(s_j)+$$

$$(-C(\alpha+r)-\frac{\alpha C_2}{r}(e^{r(t_{j+1}-s_j)}-1)+\alpha C_3 e^{r(t_{j+1}-s_j)}+C_2)e^{-rt_{j+1}}e^{-\alpha(t_{j+1}-s_j)}f(s_j) \quad j=1,2,\dots,n,$$

$$H_{2k+1,2j}=H_{2j,2k+1}=\frac{\partial^2 TC}{\partial t_{k+1}\partial s_j}=-(C_1+C(r+\theta))\,e^{\theta(s_j-t_{k+1})}f(s_j) \quad k=0, 2,\dots, n-1, j=k+1,$$

$$H_{2j+1,2k}=H_{2k,2j+1}=\frac{\partial^2 TC}{\partial s_k \partial t_{j+1}}=$$

$$-(-C(\alpha+r)-\frac{\alpha C_2}{r}(e^{r(t_{j+1}-s_j)}-1)+\alpha C_3 e^{r(t_{j+1}-s_j)}+C_2)e^{-rt_{j+1}}e^{-\alpha(t_{j+1}-s_j)}]f(s_j) \quad k=1, \ldots, n-1, \quad j=k.$$

We observe that $H_{2j+1,2j+1}>0$, $H_{2j,2j}>0$, $H_{2k+1,2j}<0$ and $H_{2k,2j+1}<0$

Let M_k be the principal minor of order k, then

$$M_1=H_{1,1}> Ar\frac{f'(t_j)}{f(t_j)}+$$

$$(C(\theta+r)+C_1)e^{\theta(s_j-t_j)}f(s_j)+(-(\alpha+r)C-\frac{\alpha C_2}{r}(e^{r(t_j-s_{j-1})}-1)-C_2+\alpha C_3 e^{r(t_j-s_{j-1})})e^{-\alpha(t_j-s_{j-1})}f(s_{j-1})$$

>0

This follows using the inequality (20).

$$M_2=\frac{\partial^2 TC}{\partial s_1^2}\frac{\partial^2 TC}{\partial t_1^2}-\frac{\partial^2 TC}{\partial s_1 \partial t_1}\frac{\partial^2 TC}{\partial t_1 \partial s_1}>$$

$$M_1\ (-(\alpha+r)C-\frac{\alpha C_2}{r}(e^{r(t_1-s_0)}-1)-C_2+\alpha C_3 e^{r(t_1-s_0)})e^{-\alpha(t_1-s_0)}f(s_0)\ +$$

$$(C(\theta+r)+C_1)e^{\theta(s_1-t_1)}f(s_1)(-(\alpha+r)C-\frac{\alpha C_2}{r}(e^{r(t_1-s_0)}-1)-C_2+\alpha C_3 e^{r(t_1-s_0)})e^{-\alpha(t_1-s_0)}f(s_0)>0$$

or, equivalently

$$M_2+\frac{\partial^2 TC}{\partial t_2 \partial s_1}M_1>$$

$$(C(\theta+r)+C_1)e^{\theta(s_1-t_1)}f(s_1)(-(\alpha+r)C-\frac{\alpha C_2}{r}(e^{r(t_1-s_0)}-1)-C_2+\alpha C_3 e^{r(t_1-s_0)})e^{-\alpha(t_1-s_0)}f(s_0)>0$$

So we have shown that $M_1>0$, $M_2>0$ and $M_2+\frac{\partial^2 TC}{\partial t_2 \partial s_1}M_1>0$

It is not difficult to verify that the principal minors of higher order satisfy the following recurrence relationships

$$M_{2j+1}=\frac{\partial^2 TC}{\partial t_{j+1}^2}M_{2j}-\left(\frac{\partial^2 TC}{\partial t_{j+1}\partial s_j}\right)^2 M_{2j-1},\ j=1,2,\ldots,n-1, \tag{A1}$$

$$M_{2j}=\frac{\partial^2 TC}{\partial s_j^2}\,M_{2j-1}-\left(\frac{\partial^2 TC}{\partial t_j \partial s_j}\right)^2 M_{2j-2},\ j=2,\ldots,n. \tag{A2}$$

Also we observe that

$$\frac{\partial^2 TC}{\partial t_{j+1}^2} > -\frac{\partial^2 TC}{\partial t_{j+1}\partial s_j}-\frac{\partial^2 TC}{\partial t_{j+1}\partial s_{j+1}} \text{ and } \frac{\partial^2 TC}{\partial s_{j+1}^2} = -\frac{\partial^2 TC}{\partial t_{j+1}\partial s_j}-\frac{\partial^2 TC}{\partial t_{j+1}\partial s_{j+1}}.$$

From (A1) and (A2) we obtain

$$M_{2j+1}+\frac{\partial^2 TC}{\partial t_{j-1}\partial s_{j-1}}\,M_{2j} > -\frac{\partial^2 TC}{\partial t_{j+1}\partial s_j}\left(M_{2j}+\frac{\partial^2 TC}{\partial t_{j+1}\partial s_j}\,M_{2j-1}\right), \tag{A3}$$

$$M_{2j}+\frac{\partial^2 TC}{\partial t_{j+1}\partial s_j}\,M_{2j-1} = -\frac{\partial^2 TC}{\partial t_j \partial s_j}\left(M_{2j-1}+\frac{\partial^2 TC}{\partial t_j \partial s_j}\,M_{2j-2}\right). \tag{A4}$$

Now, by applying inequality (A3) for j=1, we get $M_3>0$. This result, accompanied by equation (A4) for j=2 results to $M_4>0$. Using induction we can prove that the principal minors of any order are positive.

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