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EVALUATION OF ROUTES IN AN UNFRIENDLY ENVIRONMENT

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Abstract. In this paper the hard problem of locating routes on the plane is considered. The problem regards the transportation of protected materials or goods from an origin to a destination point on the plane with transportation route having to pass through an unfriendly area. The term "unfriendly" regards a fixed number of points on the plane, thereon called "damage sources", that represent installations or center points of activities that may create damage to these goods or materials. The theoretical results introduced form the basis for the application of a "branch-bound" methodology for the solution of this problem for a single damage source.

Keywords: Routing, Branch and Bounds Algorithm

1. INTRODUCTION

The problem of locating transportation routes for the transportation of hazardous materials closely interacts with that of locating obnoxious facilities (eg. chemical plants, nuclear power stations, amunition plants). Indeed, transporting hazardous materials requires, in general, facilities for processing and storing them either at the origin or at the destination.

Boffey and Karkazis (1990) considered the problem of transporting hazardous materials in a network and proposed a general methodological approach to the problem. Karkazis and Boffey (1991) presented a branch-bound algorithm for locating transportation routes in a network between two fixed points, termed origin and destination. The routes are selected so as to minimize the expected damage effects on the population in the case of an accident, under the influence of meteorological conditions.

In this paper the hard problem of locating routes on the plane is considered. The problem regards the transportation of goods or materials from a point 0 (origin) to a point D (destination) on the plane with transportation route having to pass through an unfriendly area. The term "unfriendly" regards a fixed number of points on the plane (C_i, $i = 1,..,n$) that represent installations or center points of activities that may create damage to these goods or materials. These points will be called thereon "damage sources". Possible applications may involve flights of planes over an alient area where damage sources may represent sites of enemy missile batteries.

Function $f_i(r_p)$ describes the accident probability density at a point P on the plane related to damage source at C_i. $f_i(r_p)r_p$ is assumed to be a strictly decreasing function of the euclidean distance $r_p = d(C_i, P)$ between P and damage source at C_i, that is:

$$df_i(r_p)r_p / dr_p < 0 \ \forall \ P \in s \quad \text{where s is any route from O to D} \tag{1}$$

Notice that pair (r_P, θ_P) expresses the polar coordinates of point P (see figure 1).

The accident probability $F_i(s)$ along route s confined by points 0 and D is given by the following line integral:

$$F_i(s) = \int_{P \in s} f_i(r_p) dr_p \tag{2}$$

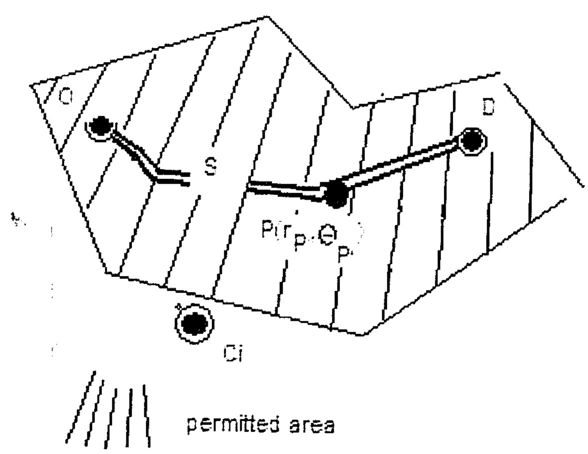

Figure l. Statement of the problem

The expected damage effects $D_i(s)$ along route s due to a damage source at C_i are defined as follows:

$$D_i(s) = w_i F_i(s) \tag{3}$$

where w_i is a weighting coefficient associated with the damage source at C_i. The expected total damage effects D(s) along route s, due to the system of the n damage sources, is given by the formula:

$$D(s) = \sum_{i=1}^{n} w_i F_i(s) \tag{4}$$

The generalized problem introduced in this paper regards the determination of a route s = s(O, D) (confined by two fixed points O and D) so as D(s) is minimised:

$$\min_{s \in R} D(s) \tag{5}$$

where R represents an area open to transportation routes characterised as "permitted area".

Notice that the constraint (l) imposed upon functions f_i is a very broad one and offers the means of representing with sufficient realism situations where the "carrier" of damage effects (e.g. missiles) follows an aerial route along which the accident probability is decreasing as the distance of the carrier from the damage source is increasing. Notice, also, that the family of functions $f_i(r) = a/r^k$ (a:constant, $k \geq 1$) satisfies the imposed constraint.

The problem that will be tackled in this paper regards the determination of routes minimising the accident probability (expected damage effects) in the presence of a single damage source:

$$\min_{s \in R} F_i(s) \tag{6}$$

2. THE THEORETICAL BACKGROUND OF THE PROBLEM

If a point P on the plane is represented by polar coordinates (r_p, θ_p) with respect to damage source C_i then a route s=s(O, D) could be expressed in the form of a "radial" function $s(\theta)$ $\theta_O \leq \theta \leq \theta_D$, where θ_O, θ_D are the angular coordinates of O, D respectively. Note that, for a given point P, if $\theta = \theta_p$,then $s(\theta) = r_p$). The following well established result, regarding line integrals, gives the means of expressing accident probability along a route s in terms of an ordinary integral.

Result 1. The accident probability along a route s = s (O, D) with respect to a damage source C_i is given by the following formula:

$$F_i(s) = \int_{\grave{e}_O}^{\grave{e}_D} f_i(s(\grave{e}))\sqrt{[s'(\grave{e})]^2 + [s(\grave{e})]^2}\ d\grave{e} \tag{7}$$

where $s(\theta)$ is the radial function, with respect to damage source $C_{i,}$ representing path s.

Result 2. The expected total damage effects along route s = s(O, D) are given by the following formula:

$$D(s) = \int_{\theta_O}^{\theta_D} \left(\Sigma_{i=1} w_i f_i(s(\theta))\sqrt{[s'(\theta)]^2 + [s(\theta)]^2}\right) d\theta$$

Proof. Immediate, from Result 1 and formula (4).

Let $s(\theta)$, $\theta \in [\theta 1, \theta 2]$ be the radial function describing a route s with respect to a damage source C_i. We define the following two basic types of routes:

Definition 1. If $s(\theta)$ =constant $\forall\ \theta \in [\theta 1, \theta 2]$ then s will be called **"polar"** (or "circular") route with respect to damage source C_i (route s_1 of figure 2) whereas if s is part of a straight line passing through C_i then s will be called **"radial"** route with respect to damage source C_i (route s_2 of figure 2).

Result3. The optimal route for problems (5) and (6) does not contain loops.

Proof. It is an immediate consequence of the fact that the D(s) (respectively $F_i(s)$) strictly increases with the length of route s.

Definition 2. A route will be called "simple" if it does not contain loops.

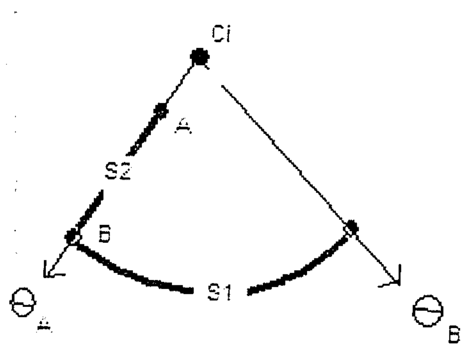

Figure 2. Radial and polar routes

Theorem 1. Let AC_iB be an angle centered on C_i and s a simple polar route with respect to C_i confined by points A, B that belong to the two sides C_iA, C_iB of the angle respectively (see figure 3). Let also q be a route such that its end points A* and B* belong to the sides C_iA, C_iB of the angle respectively, whereas its interior points lie in the interior of the area enclosed by the sides of the angle and polar route s. Then

$$F_i(s) < F_i(q)$$

Proof. Let $s(\theta)$, $q(\theta)$ be the radial functions describing routes s and q respectively with respect to C_i ($\theta \varepsilon [\theta_A, \theta_B]$). Then from the statement of the theorem the following are immediate:

$$s(è) > q(è) \ \forall \ è \in (è_{Á}, è_{Â}) \tag{8}$$

and also

$$s'(è) = 0 \ \forall \ è \in [è_{Á}, è_{Â}] \tag{9}$$

Furthermore, from (9) and result 1 we get that

$$F_i(s) = \int_{è_{Á}}^{è_{Â}} f_i(s(è))s(è)dè \tag{10}$$

On the other hand from result 1

$$F_i(q) = \int_{è_{Á}}^{è_{Â}} f_i(q(è))\sqrt{[q'(è)]^2 + [q(è)]^2}\ dè$$ and thus

$$F_i(q) \geq K_i(q) = \int_{è_{Á}}^{è_{Â}} f_i(q(è))q(è)dè \tag{11}$$

From the assumption (1) and the relation (8) it is immediate that

$$K_i(q) > F_i(s) \tag{12}$$

Relations (11) and (12) establish the validity of the theorem.

Result4. If $s=s(\theta)$, $\theta \in [\theta_A,\theta_B]$ represents a polar route with respect to C_i then (see figure 2):

$$F_i(s) = f_i(r_0) r_0 (\grave{e}_{\hat{A}} - \grave{e}_{\acute{A}}) \text{ where } r_0 = s(\grave{e}_{\acute{A}})$$

Proof. Directly from Result 1 and relation $s'(\grave{e}) = 0$ which holds for polar routes.

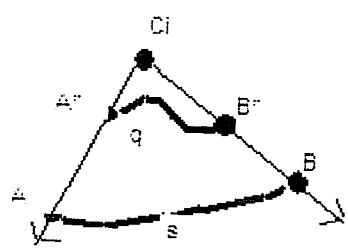

Figure 3. Theorem 1

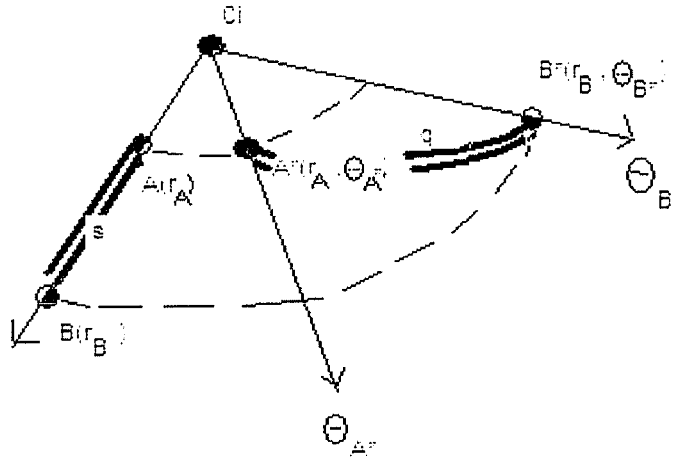

Figure 4. Theorem 2

Theorem 2. Let s be a radial route with respect to Ci confined by points A and B such that $s_B > s_A$ where $s_B = d(C_i, B)$ and $s_A = d(C_i, A)$ (see figure 4). If q is any route (different than s) described by the radial function $q(\theta)$ $\grave{e} \in [\grave{e}_{A'}, \grave{e}_{A'}]$ with respect to C_i such that $q(\grave{e}_{A'}) = s_A$ and $q(\grave{e}_{A'}) = s_B$ then

$$F_i(q) \geq F_i(s)$$

Proof. From result 1 we have

$$F_i(q) = \int_{\grave{e}_{A'}}^{\grave{e}_{A'}} f_i(q(\grave{e}))\sqrt{[q'(\grave{e})]^2 + [q(\grave{e})]^2}\, d\grave{e} \text{ hence}$$

$$F_i(q) \geq \int_{\grave{e}_{A'}}^{\grave{e}_{A'}} f_i(q(\grave{e}))q'(\grave{e})d\grave{e} = \int_{\grave{e}_{A'}}^{\grave{e}_{A'}} f_i(q(\grave{e}))dq(\grave{e}) \quad (13)$$

If we set, next, $r=q(\theta)$ then it is obvious that

$$\int_{\grave{e}_{A'}}^{\grave{e}_{A'}} f_i(q(\grave{e}))dq(\grave{e}) = \int_{r_0}^{r_B} f_i(r)dr = F_i(s)$$

(13a)

From (13) and (13a)we get that

$$F_i(q) \geq F_i(s)$$

The above relation establishes the validity of the result.

3. SOME FEATURES OF THE OPTIMAL SOLUTION OF THE SINGLE DAMAGE SOURCE PROBLEM

The following theorem establishes an important characteristic of the optimal route s* of problem (6), namely that the polar function $s^*(\theta)$ describing s* is convex with respect to C_i.

This characteristic is a necessary requirement for establishing efficient algorithmic solution methods converging to the optimum.

Theorem 3. The polar function $s^*(\theta)$, $\theta \in [\theta_O, \theta_D]$ with respect to C_i expressing the optimal (route) solution of problem (6) is convex with respect to C_i.

Proof. Let's assume for the moment that $s^*(\theta)$ possesses a concave part Then this part of the route will lie in the interior of an area confined by the angle OC_iD and a polar path, say p, with respect to C_i. Let A, B be the intersections of p with this concave part of s*. Set, next, $s_l=s^*(A, B)$ and $p_l=p(A, B)$ (figure 5). Then from theorem 1 we get that

$$F_i(s_l) > F_i(p_l)$$

and thus the substitution of s_l with the polar path p_l would decrease the accident probability of s*, a fact that contradicts the optimality of s*.

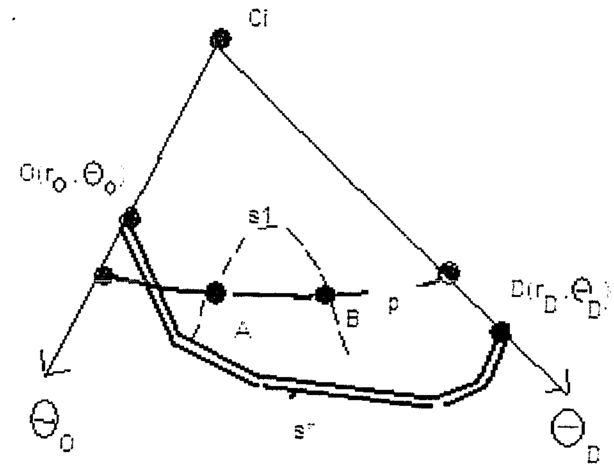

Figure 5.Theorem3

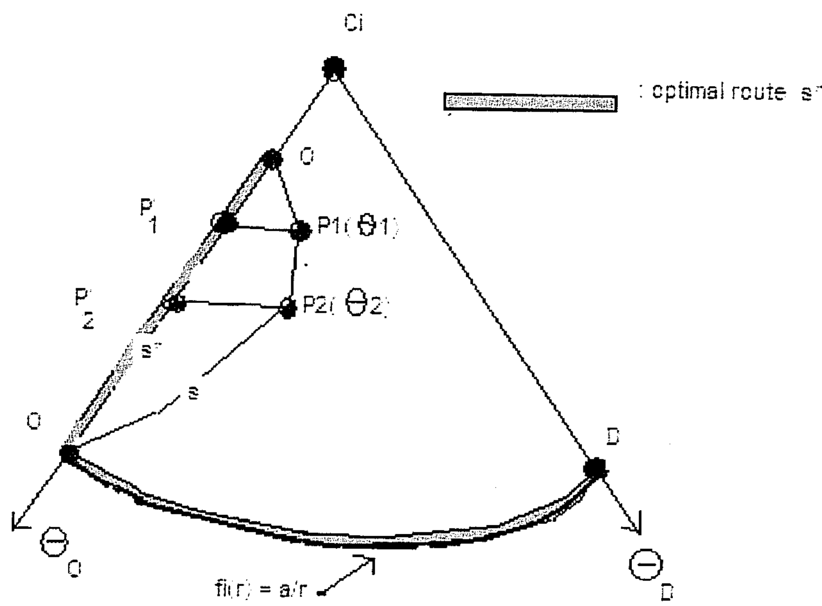

Figure 6. Optimal routes for special cases

Based on the theoretical background established in section 2 we could derive the optimal route of problem (6) in the following two special cases:

i. when origin and destination are equidistant from the damage source at C_i and the accident probability function takes the form $f_i(r) = a/r$ where a is a constant

ii. when origin and destination lie on the same radial line (see theorem 5)

Theorem 4. If the origin O and destination D are equidistant from C_i and the accident probability function f_i with respect to C_i takes the form:

$f_i(r) = a/r$, a: constant

then the optimal route s* from O to D (figure 6) is the polar route with respect to C_i confined by points O and D.

Proof. Consider a simple route $s = s(\theta)$, $\theta \in [\theta_O, \theta_D]$ confined by points O and D. Then from result 2

$$F_i(s) = \int_{\theta_O}^{\theta_D} f_i(s(\theta))\sqrt{[s'(\theta)]^2 + [s(\theta)]^2}\, d\theta \qquad (14)$$

From (14) it is apparent that

$$F_i(s) \geq \int_{\theta_O}^{\theta_D} f_i(s(\theta))s(\theta)d\theta = \int_{\theta_O}^{\theta_D} (a / s(\theta))s(\theta)d\theta = a(\theta_D - \theta_O) \qquad (15)$$

and hence

$$F_i(s) \geq a(\theta_D - \theta_O) \tag{16}$$

Consider next the polar route p connecting O and D; it is obvious that

$$p(\theta) = \text{constant and } p'(\theta) = 0 \text{ consequently} \tag{17}$$

$$F_i(p) = \int_{\theta_O}^{\theta_D} (a / p(\theta))\sqrt{[p'(\theta)]^2 + [p(\theta)]^2}\, d\theta = \int_{\theta_O}^{\theta_D} (a / p(\theta))p(\theta) = a(\theta_D - \theta_O) \tag{18}$$

From (16) and (18) we get that

$$F_i(s) \geq F_i(p)$$

Theorem 5. If the origin O and destination D lie on a straight line passing through Ci then the optimal route s* from O to D is the radial (linear) route connecting O and D.

Proof. The proof is a straightforward consequence of theorem 2. Consider a route $s \in R$. Routes can be divided into a number of segments each one of them capable of being expressed by a radial function $s(\theta)$ with θ appropriately confined (see figure 6). Let $s(P_1,P_2)$ be such a segment of s confined by points P_1 and P_2 with $\theta \in [\theta_1,\theta_2]$. Consider next the part $s^*(P_1', P_2')$ of s* that satisfies the relations:

$d(C_i, P_1') = d(C_i, P_1)$ and $d(C_i, P_2') = d(C_i, P_2)$

Then, for the routes $s(P_1, P_2)$ and $s^*(P_1', P_2')$ theorem 2 states:

$$F_i(s^*(P_1', P_2')) \leq F_i(s(P_1, P_2)) \tag{19}$$

Applying formula (19) for all segments of the above division we get that

$$F_i(s^*) \leq F_i(s) \tag{20}$$

Realation (20) establishes the validity of the theorem.

For the rest of this section it is assumed without loss of generality that

$$d(C_i, O) \leq d(C_i, D) \tag{21}$$

It is proved, next, that under the broad condition (1) imposed on fi the initial area of search for an optimal solution route s* of the problem (6) can be reduced to a polar trapezoid confined by the radial lines C_iO and C_iD and two polar (cyclical) lines cantered at Ci. The first of them passes through origin O whereas the second through a point $U \in C_iD$ such that $d(C_i,U) \geq d(C_i,D)$. Finally, using the two basic types of routes (radial and polar) lower and upper bounds for the optimal value $F_i(s^*)$ of problem (6) are established.

Theorem 6. If the function f_i satisfies the constraint (I) then the optimal route s* from $O(r_0, \theta_0)$ to $D(r_D,\theta_D)$ lies in the polar trapezium confined by the radial lines C_iO and C_iD and the polar routes (with respect to C_i) passing through points O and $U(r_U,\theta_U)$ where U is a point on the radial line C_iD satisfying the constraint:

$$F_i(r(D,U)) = 0.5F_i(p(D',D))$$

Note that r(D, U) is the radial route confined ,by points D and U and p(D', D) is the polar route confined by points D' and D $(D' \in C_iO : d(C_i, D') = r_D$, see figure 7).

Proof. From Theorem 1 it is immediate that s* cannot intersect the polar triangle C_iOO' where O' is the intersection of the polar route passing through O with radial line C_iD. Furthermore, from Theorem 2 it is immediate that s* cannot intersect the area extending outside the angle OC_iD. Finally, we will prove that s* cannot intersect the area extending outside the circle centered at C_i and having radius equal to $d(C_i ,U)$. We assume for the moment that s* intersects it. Let Z be the most distant from C_i point of s*. Set $r_z=d(C_i, Z)$ and consider the following two sub-routes of s*: $s_1=s_1(M, Z)$ and $s_2 =s_2(Z, D)$ where M is the intersection of s* with the polar route p(D′,D). Then from Theorem 2 we get that

$$F_i(s_1) \geq F_i(r(U,Z')) > F_i(r(U,D)) \text{ and } F_i(s_2) \geq F_i(r(U,Z)) > F_i(r(O,D)) \quad (22)$$

where Z' is the intersection of the polar route passing through Z with the radial line C_iD and r(U, Z') is the radial route confined by points U and Z'.

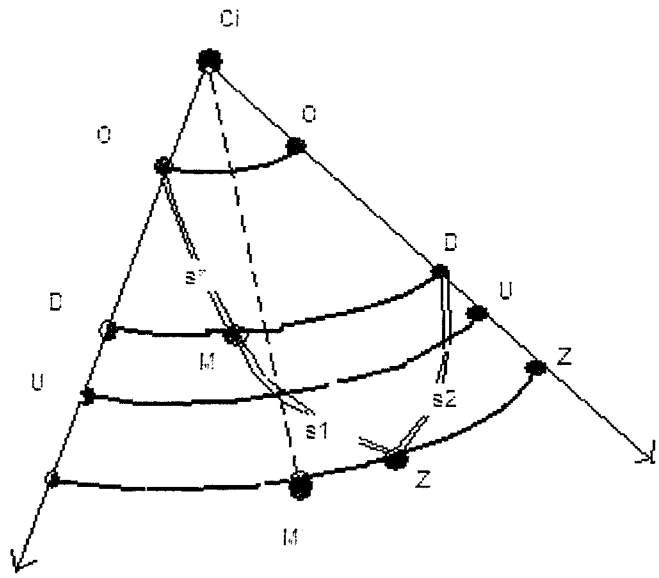

Figure 7. Theorem 6

From (22) we have that

$$F_i(s_1 \cup s_2) > 2F_i(r(U,D))$$

On the other hand, from the statement of the theorem we get that

$$F_i(r(U,D)) = 0.5F_i(p(D',D)) \quad (23)$$

From (22) and (23) it is immediate that

$$F_i(s_1 \cup s_2) > F_i(p(M,D))$$

That is, by substituting route $s = s_1 \cup s_2$ with polar route p(M, D) we could further decrease the value of F_i. The last conclusion contradicts the optimality of s* and thus the original assumption (that s* extends outside circle $C(C_i, r_u)$) is false.

Theorem 7. The accident probability Fi(s*) of the optimal route s* from O to D satisfies the following bounding relations:

$$F_i(r(O,D')) + F_i(p(D',D) \geq F_i(s^*) \geq F_i(r(O',D)) + 0.5F_i(p(D',D))$$

where $r(O,D')$ is the radial route confined by points O and D' and p(D', D) is the polar route confined by D' and D.

Proof. Consider the route $s = r(O,D') \cup p(D',D)$ confined by O and D. Then the optimality of s* requires that

$$F_i(s) \geq F_i(s^*)$$

and hence the left part of the relation is valid.

Furthermore, as a consequence of theorem 6 (see figure 7) s* belongs to the polar triangle C_iUU' where $U(r_u, \theta_u)$ satisfies the following relation:

$$F_i(r(D,U)) = \int_{r_0}^{r_U} f_i(r)dr = 0,5F_i(p(O',D)) \qquad (24)$$

From theorem 2 and relation (24) we get that

$$F_i(s^*) \geq F_i(r(O', U)) = F_i(r(O', D)) + F_i(r(D, U)) \tag{25}$$

where r(X, Y) represents the radial route from X to Y.

From formuli (24) and (25, it is immediate that

$$F_i(s^*) \geq F_i(r(O', D)) + 0.5F_i(p(D', D))$$

which proves the right part of the relation in the statement of the theorem.

4. CONCLUSIONS

In the previous sections a theoretical basis was established enabling the application of very powerful algorithms, such as "Branch-Bound", which are capable of solving optimally the problem. The paper introduces a spatial division of the plane into polar trapezoids and a fathoming process based on the bounding constraints developed in section 3. It is evident that the above methodology can be easily generalized to similar problems in the 3-dimensional space such as air transport and cruise missile route evaluations.

REFERENCES

1. **Boffey, B. and Karkazis, J. (1990),** "Routing of hazardous materials", *Proceedings Meeting V of the Euro Working Group on Locational Analysis,* Facultes Universitaire Notre-Dame de la Paix, Namur (Belgium)

2. **Karkazis, J. and Papadimitriou, C.(1992),** "A branch-and-bound algorithm for the location of facilities causing atmospheric pollution", *European Journal of Operation., Research* (to appear)

3. **Karkazis, J. and Boffey, B. (1995),** "Optimal location of routes for vehicles transporting hazardous materials", *European Journal of Operational Research,* Vol.86, pp. 201-215