EVALUATION OF ROUTES IN AN UNFRIENDLY ENVIRONMENT

John Karkazis

Department of Shipping, Transport and Trade
University of the Aegean
Chios, Greece

Abstract. In this paper the hard problem of locating routes on the plane is considered. The problem regards the transportation of protected materials or goods from an origin to a destination point on the plane with transportation route having to pass through an unfriendly area. The term "unfriendly" regards a fixed number of points on the plane, thereon called "damage sources", that represent installations or center points of activities that may create damage to these goods or materials. The theoretical results introduced form the basis for the application of a "branch-bound" methodology for the solution of this problem for a single damage source.

Keywords: Routing, Branch and Bounds Algorithm
1. INTRODUCTION

The problem of locating transportation routes for the transportation of hazardous materials closely interacts with that of locating obnoxious facilities (e.g., chemical plants, nuclear power stations, ammunition plants). Indeed, transporting hazardous materials requires, in general, facilities for processing and storing them either at the origin or at the destination.

Boffey and Karkazis (1990) considered the problem of transporting hazardous materials in a network and proposed a general methodological approach to the problem. Karkazis and Boffey (1991) presented a branch-bound algorithm for locating transportation routes in a network between two fixed points, termed origin and destination. The routes are selected so as to minimize the expected damage effects on the population in the case of an accident, under the influence of meteorological conditions.

In this paper the hard problem of locating routes on the plane is considered. The problem regards the transportation of goods or materials from a point 0 (origin) to a point D (destination) on the plane with transportation route having to pass through an unfriendly area. The term "unfriendly" regards a fixed number of points on the plane (C_i, i = 1,..n) that represent installations or center points of activities that may create damage to these goods or materials. These points will be called thereon "damage sources". Possible applications may involve flights of planes over an alien area where damage sources may represent sites of enemy missile batteries.

Function f_i(r_p) describes the accident probability density at a point P on the plane related to damage source at C_i. f_i(r_p)r_p is assumed to
be a strictly decreasing function of the euclidean distance \( r_p = d(C_i, P) \) between P and damage source at \( C_i \), that is:

\[
df_i(r_p)r_p/\mathrm{dr}_p < 0 \quad \forall \ P \in s \quad \text{where } s \text{ is any route from O to D} \tag{1}
\]

Notice that pair \((r_p, \theta_p)\) expresses the polar coordinates of point P (see figure 1).

The accident probability \( F_i(s) \) along route \( s \) confined by points 0 and D is given by the following line integral:

\[
F_i(s) = \int_{r_p \in \Omega} f_i(r_p) \mathrm{dr}_p \tag{2}
\]

![Figure 1. Statement of the problem](image)

The expected damage effects \( D_i(s) \) along route \( s \) due to a damage source at \( C_i \) are defined as follows:
\[ D_i(s) = w_i F_i(s) \] (3)

where \( w_i \) is a weighting coefficient associated with the damage source at \( C_i \). The expected total damage effects \( D(s) \) along route \( s \), due to the system of the \( n \) damage sources, is given by the formula:

\[ D(s) = \sum_{i=1}^{n} w_i F_i(s) \] (4)

The generalized problem introduced in this paper regards the determination of a route \( s = s(O, D) \) (confined by two fixed points \( O \) and \( D \)) so as \( D(s) \) is minimised:

\[ \min_{s \in R} D(s) \] (5)

where \( R \) represents an area open to transportation routes characterised as "permitted area".

Notice that the constraint (I) imposed upon functions \( f_i \) is a very broad one and offers the means of representing with sufficient realism situations where the "carrier" of damage effects (e.g. missiles) follows an aerial route along which the accident probability is decreasing as the distance of the carrier from the damage source is increasing. Notice, also, that the family of functions \( f_i(r) = a / r^k \) (\( a \):constant, \( k \geq 1 \)) satisfies the imposed constraint.
The problem that will be tackled in this paper regards the determination of routes minimising the accident probability (expected damage effects) in the presence of a single damage source:

\[ \min_{s \in R} F_i(s) \]  

(6)

2. THE THEORETICAL BACKGROUND OF THE PROBLEM

If a point P on the plane is represented by polar coordinates \((r_p, \theta_p)\) with respect to damage source \(C_i\) then a route \(s=s(O, D)\) could be expressed in the form of a "radial" function \(s(\theta) \theta_0 \leq \theta \leq \theta_D\), where \(\theta_O, \theta_D\) are the angular coordinates of O, D respectively. Note that, for a given point P, if \(\theta = \theta_p\), then \(s(\theta) = r_p\). The following well established result, regarding line integrals, gives the means of expressing accident probability along a route \(s\) in terms of an ordinary integral.

**Result 1.** The accident probability along a route \(s = s(O, D)\) with respect to a damage source \(C_i\) is given by the following formula:

\[ F_i(s) = \int_{\theta_0}^{\theta_D} f_i(s(\theta)) \sqrt{[s'(\theta)]^2 + [s(\theta)]^2} \, d\theta \]  

(7)

where \(s(\theta)\) is the radial function, with respect to damage source \(C_i\), representing path \(s\).

**Result 2.** The expected total damage effects along route \(s = s(O, D)\) are given by the following formula:
\[ D(s) = \int_{0}^{\hat{\theta}_0} \left( \hat{\theta} \right) w_i \left( s(\hat{\theta}) \right) \sqrt{[s'(\hat{\theta})]^2 + [s(\hat{\theta})]^2} \, d\hat{\theta} \]

**Proof.** Immediate, from Result 1 and formula (4).

Let \( s(\theta) \), \( \theta \in [\theta_1, \theta_2] \) be the radial function describing a route \( s \) with respect to a damage source \( C_i \). We define the following two basic types of routes:

**Definition 1.** If \( s(\theta) = \text{constant} \ \forall \ \theta \in [\theta_1, \theta_2] \) then \( s \) will be called "polar" (or "circular") route with respect to damage source \( C_i \) (route \( s_1 \) of figure 2) whereas if \( s \) is part of a straight line passing through \( C_i \) then \( s \) will be called "radial" route with respect to damage source \( C_i \) (route \( s_2 \) of figure 2).

**Result 3.** The optimal route for problems (5) and (6) does not contain loops.

**Proof.** It is an immediate consequence of the fact that the \( D(s) \) (respectively \( F_i(s) \)) strictly increases with the length of route \( s \).

**Definition 2.** A route will be called "simple" if it does not contain loops.

---

**Figure 2. Radial and polar routes**
Theorem 1. Let ACiB be an angle centered on C; and s a simple polar route with respect to C; confined by points A, B that belong to the two sides C;A, C;B of the angle respectively (see figure 3). Let also q be a route such that its end points A* and B* belong to the sides C;A, C;B of the angle respectively, whereas its interior points lie in the interior of the area enclosed by the sides of the angle and polar route s. Then

\[ F_i(s) < F_i(q) \]

Proof. Let \( s(\theta), q(\theta) \) be the radial functions describing routes s and q respectively with respect to C; (\( \theta \in [\theta_A, \theta_B] \)). Then from the statement of the theorem the following are immediate:

\[ s(\hat{\theta}) > q(\hat{\theta}) \quad \forall \quad \hat{\theta} \in (\hat{\theta}_A, \hat{\theta}_B) \]  

(8)

and also

\[ s'(\hat{\theta}) = 0 \quad \forall \quad \hat{\theta} \in [\hat{\theta}_A, \hat{\theta}_B] \]  

(9)

Furthermore, from (9) and result 1 we get that

\[ F_i(s) = \int_{\hat{\theta}_A}^{\hat{\theta}_B} f_i(s(\hat{\theta}))s(\hat{\theta})d\hat{\theta} \]  

(10)

On the other hand from result 1

\[ F_i(q) = \int_{\hat{\theta}_A}^{\hat{\theta}_B} f_i(q(\hat{\theta}))\sqrt{[q'(\hat{\theta})]^2 + [q(\hat{\theta})]^2} \ \ d\hat{\theta} \]  

and thus

\[ F_i(q) \geq K_i(q) = \int_{\hat{\theta}_A}^{\hat{\theta}_B} f_i(q(\hat{\theta}))q(\hat{\theta})d\hat{\theta} \]  

(11)
From the assumption (1) and the relation (8) it is immediate that

$$K_i(q) > F_i(s)$$  \hfill (12)

Relations (11) and (12) establish the validity of the theorem.

**Result 4.** If $s = s(\theta)$, $\theta \in [\theta_A, \theta_B]$ represents a polar route with respect to $C_i$ then (see figure 2):

$$F_i(s) = f_i(r_0) r_0 (\dot{e}_A - \dot{e}_A)$$ where $r_0 = s(\dot{e}_A)$

**Proof.** Directly from Result 1 and relation $s'(\dot{e}) = 0$ which holds for polar routes.

![Figure 3. Theorem 1](image-url)
Figure 4. Theorem 2

**Theorem 2.** Let \( s \) be a radial route with respect to \( C_i \) confined by points A and B such that \( s_B > s_A \) where \( s_B = d(C_i, B) \) and \( s_A = d(C_i, A) \) (see figure 4). If \( q \) is any route (different than \( s \)) described by the radial function \( q(\theta) \in \epsilon_{\epsilon_{A^*}} \) with respect to \( C_i \) such that \( q(\epsilon_{A^*}) = s_A \) and \( q(\epsilon_{A^*}) = s_B \) then

\[
F_i(q) \geq F_i(s)
\]

**Proof.** From result 1 we have

\[
F_i(q) = \int_{\epsilon_{A^*}}^{\hat{\epsilon}_{A^*}} f_i(q(\hat{\epsilon})) \sqrt{q'(\hat{\epsilon})^2 + [q(\hat{\epsilon})]^2} \, d\hat{\epsilon}
\]

Hence

\[
F_i(q) \approx \int_{\epsilon_{A^*}}^{\hat{\epsilon}_{A^*}} f_i(q(\hat{\epsilon}))q'(\hat{\epsilon}) \, d\hat{\epsilon} = \int_{\epsilon_{A^*}}^{\hat{\epsilon}_{A^*}} f_i(q(\hat{\epsilon})) \, dq(\hat{\epsilon})
\]

(13)

If we set, next, \( r = q(\theta) \) then it is obvious that

\[
\int_{\epsilon_{A^*}}^{\hat{\epsilon}_{A^*}} f_i(q(\hat{\epsilon})) \, dq(\hat{\epsilon}) = \int_{\epsilon_{A^*}}^{\hat{\epsilon}_{A^*}} f_i(r) \, dr = F_i(s)
\]

(13a)

From (13) and (13a) we get that

\[
F_i(q) \geq F_i(s)
\]

The above relation establishes the validity of the result.
3. **SOME FEATURES OF THE OPTIMAL SOLUTION OF THE SINGLE DAMAGE SOURCE PROBLEM**

The following theorem establishes an important characteristic of the optimal route $s^*$ of problem (6), namely that the polar function $s^*(\theta)$ describing $s^*$ is convex with respect to $C_i$.

This characteristic is a necessary requirement for establishing efficient algorithmic solution methods converging to the optimum.

**Theorem 3.** The polar function $s^*(\theta)$, $\theta \in [\theta_0, \theta_D]$ with respect to $C_i$ expressing the optimal (route) solution of problem (6) is convex with respect to $C_i$.

**Proof.** Let’s assume for the moment that $s^*(\theta)$ possesses a concave part. Then this part of the route will lie in the interior of an area confined by the angle $OC_iD$ and a polar path, say $p$, with respect to $C_i$. Let $A, B$ be the intersections of $p$ with this concave part of $s^*$. Set, next, $s_i = s^*(A, B)$ and $p_i = p(A, B)$ (figure 5). Then from theorem 1 we get that

$$F_i(s_i) > F_i(p_i)$$

and thus the substitution of $s_i$ with the polar path $p_i$ would decrease the accident probability of $s^*$, a fact that contradicts the optimality of $s^*$. 
Figure 5. Theorem 3

Figure 6. Optimal routes for special cases
Based on the theoretical background established in section 2 we could derive the optimal route of problem (6) in the following two special cases:

i. when origin and destination are equidistant from the damage source at $C_i$ and the accident probability function takes the form $f_i(r) = \frac{a}{r}$ where $a$ is a constant

ii. when origin and destination lie on the same radial line (see theorem 5)

**Theorem 4.** If the origin $O$ and destination $D$ are equidistant from $C_i$ and the accident probability function $f_i$ with respect to $C_i$ takes the form:

$$f_i(r) = \frac{a}{r}, \ a: \text{constant}$$

then the optimal route $s^*$ from $O$ to $D$ (figure 6) is the polar route with respect to $C_i$ confined by points $O$ and $D$.

**Proof.** Consider a simple route $s = s(\theta)$, $\theta \in [\theta_0, \theta_D]$ confined by points $O$ and $D$. Then from result 2

$$F_i(s) = \int_{\theta_0}^{\theta_D} f_i(s(\theta))\sqrt{[s'(\theta)]^2 + [s(\theta)]^2} \ d\theta$$

(14)

From (14) it is apparent that

$$F_i(s) \geq \int_{\theta_0}^{\theta_D} f_i(s(\theta))s(\theta) \ d\theta = \int_{\theta_0}^{\theta_D} (a / s(\theta))s(\theta) \ d\theta = a(\theta_D - \theta_0)$$

(15)
and hence

\[ F_i(s) \geq a(\dot{e}_D - \dot{e}_O) \]  \hspace{1cm} (16)

Consider next the polar route \( p \) connecting \( O \) and \( D \); it is obvious that

\[ p(\theta) = \text{constant and } p'(\dot{e}) = 0 \] consequently

\[ F_i(p) = \int_{e_0}^{e_D} \left( \frac{a}{p(\dot{e})}\sqrt{[p'(\dot{e})]^2 + [p(\dot{e})]^2} \right) \, d\dot{e} = \int_{e_0}^{e_D} \left( \frac{a}{p(\dot{e})}p(\dot{e}) \right) \, d\dot{e} = a(\dot{e}_D - \dot{e}_O) \] \hspace{1cm} (18)

From (16) and (18) we get that

\[ F_i(s) \geq F_i(p) \]

**Theorem 5.** If the origin \( O \) and destination \( D \) lie on a straight line passing through \( C_i \) then the optimal route \( s^* \) from \( O \) to \( D \) is the radial (linear) route connecting \( O \) and \( D \).

**Proof.** The proof is a straightforward consequence of theorem 2. Consider a route \( s \in R \). Routes can be divided into a number of segments each one of them capable of being expressed by a radial function \( s(\theta) \) with \( \theta \) appropriately confined (see figure 6). Let \( s(P_1,P_2) \) be such a segment of \( s \) confined by points \( P_1 \) and \( P_2 \) with \( \theta \in [\theta_1,\theta_2] \).

Consider next the part \( s'(P_1',P_2') \) of \( s^* \) that satisfies the relations:
\[ d(C_i, P'_1) = d(C_i, P_1) \quad \text{and} \quad d(C_i, P'_2) = d(C_i, P_2) \]

Then, for the routes \( s(P_1, P_2) \) and \( s'(P'_1, P'_2) \) theorem 2 states:

\[ F_i(s'(P'_1, P'_2)) \leq F_i(s(P_1, P_2)) \]  \hspace{1cm} (19)

Applying formula (19) for all segments of the above division we get that

\[ F_i(s^*) \leq F_i(s) \]  \hspace{1cm} (20)

Relation (20) establishes the validity of the theorem.

For the rest of this section it is assumed without loss of generality that

\[ d(C_i, O) \leq d(C_i, D) \]  \hspace{1cm} (21)

It is proved, next, that under the broad condition (1) imposed on \( f_i \) the initial area of search for an optimal solution route \( s^* \) of the problem (6) can be reduced to a polar trapezoid confined by the radial lines \( C_iO \) and \( C_iD \) and two polar (cyclical) lines cantered at \( Ci \). The first of them passes through origin \( O \) whereas the second through a point \( U \in C_iD \) such that \( d(C_i, U) \geq d(C_i, D) \). Finally, using the two basic types of routes (radial and polar) lower and upper bounds for the optimal value \( F_i(s^*) \) of problem (6) are established.
**Theorem 6.** If the function $f_i$ satisfies the constraint (1) then the optimal route $s^*$ from $O(r_0, \theta_0)$ to $D(r_D, \theta_D)$ lies in the polar trapezium confined by the radial lines $C_iO$ and $C_iD$ and the polar routes (with respect to $C_i$) passing through points $O$ and $U(r_U, \theta_U)$ where $U$ is a point on the radial line $C_iD$ satisfying the constraint:

$$F_i(r(D, U)) = 0.5F_i(p(D', D))$$

Note that $r(D, U)$ is the radial route confined by points $D$ and $U$ and $p(D', D)$ is the polar route confined by points $D'$ and $D (D' \subseteq C_iO : d(C_i, D') = r_D)$, see figure 7).

**Proof.** From Theorem 1 it is immediate that $s^*$ cannot intersect the polar triangle $C_iOO'$ where $O'$ is the intersection of the polar route passing through $O$ with radial line $C_iD$. Furthermore, from Theorem 2 it is immediate that $s^*$ cannot intersect the area extending outside the angle $OC_iD$. Finally, we will prove that $s^*$ cannot intersect the area extending outside the circle centered at $C_i$ and having radius equal to $d(C_i, U)$. We assume for the moment that $s^*$ intersects it. Let $Z$ be the most distant point from $C_i$ point of $s^*$. Set $r_z = d(C_i, Z)$ and consider the following two sub-routes of $s^*$: $s_i = s_i(M, Z)$ and $s_2 = s_2(Z, D)$ where $M$ is the intersection of $s^*$ with the polar route $p(D', D)$. Then from Theorem 2 we get that

$$F_i(s_i) = F_i(r(U, Z')) > F_i(r(U, D)) \text{ and } F_i(s_2) = F_i(r(U, Z)) > F_i(r(O, D))$$

(22)
where $Z'$ is the intersection of the polar route passing through $Z$ with the radial line $C_iD$ and $r(U, Z')$ is the radial route confined by points $U$ and $Z'$.

**Figure 7. Theorem 6**

From (22) we have that

$$F_i(s_1 \cup s_2) > 2F_i(r(U, D))$$

On the other hand, from the statement of the theorem we get that

$$F_i(r(U, D)) = 0.5F_i(p(D', D)) \quad (23)$$

From (22) and (23) it is immediate that

$$F_i(s_1 \cup s_2) > F_i(p(M, D))$$
That is, by substituting route \( s = s_1 U s_2 \) with polar route \( p(M, D) \) we could further decrease the value of \( F_i \). The last conclusion contradicts the optimality of \( s^* \) and thus the original assumption (that \( s^* \) extends outside circle \( C(C_i, r_o) \)) is false.

**Theorem 7.** The accident probability \( F_i(s^*) \) of the optimal route \( s^* \) from \( O \) to \( D \) satisfies the following bounding relations:

\[
F_i(r(O, D')) + F_i(p(D', D)) \geq F_i(s^*) \geq F_i(r(O', D)) + 0.5F_i(p(D', D))
\]

where \( r(O, D') \) is the radial route confined by points \( O \) and \( D' \) and \( p(D', D) \) is the polar route confined by \( D' \) and \( D \).

**Proof.** Consider the route \( s = r(O, D') \cup p(D', D) \) confined by \( O \) and \( D \). Then the optimality of \( s^* \) requires that

\[
F_i(s) \geq F_i(s^*)
\]

and hence the left part of the relation is valid.

Furthermore, as a consequence of theorem 6 (see figure 7) \( s^* \) belongs to the polar triangle \( C_iUU' \) where \( U(r_u, \theta_u) \) satisfies the following relation:

\[
F_i(r(D, U)) = \int_{\theta_u}^{\theta_u} f_i(r)dr = 0.5F_i(p(O', D)) \quad (24)
\]

From theorem 2 and relation (24) we get that
\[ F_i(s^*) \geq F_i(r(O', U)) = F_i(r(O', D)) + F_i(r(D, U)) \] (25)

where \( r(X, Y) \) represents the radial route from \( X \) to \( Y \).

From formula (24) and (25), it is immediate that

\[ F_i(s^*) \geq F_i(r(O', D)) + 0.5F_i(p(D', D)) \]

which proves the right part of the relation in the statement of the theorem.

4. CONCLUSIONS

In the previous sections a theoretical basis was established enabling the application of very powerful algorithms, such as "Branch-Bound", which are capable of solving optimally the problem. The paper introduces a spatial division of the plane into polar trapezoids and a fathoming process based on the bounding constraints developed in section 3. It is evident that the above methodology can be easily generalized to similar problems in the 3-dimensional space such as air transport and cruise missile route evaluations.
REFERENCES

